Dijon school May 2022

LECTURE 1: GIT and good moduli spaces

Goal: Give stack-theoretic treatment of GIT

§1. Mumford's Geometric Invariant Theory (GIT)

Setup: G lin reductive alg gp / $k = \bar{k}$

$\uparrow$ functor $\mathrm{Rep}(G) \to \mathrm{Vect}_k$ is exact
$V \mapsto V^G$

Exs: - finite gps G with $\mathrm{char}(k) \nmid |G|$
- $\mathbb{G}_m$ or torus $\mathbb{G}_m^n$ in any char
- $SL_n, GL_n, \ldots$ in char $= 0$

Main thm (affine case)

Let $X = \operatorname{Spec} A$ be k-scheme with G-action.
Let $A^G \subset A$ subring of G-invariants.
Consider $\pi : \operatorname{Spec} A \to \operatorname{Spec} A^G$ (this is a G-invariant map)

1. For $Z \subseteq X$ G-invariant subscheme, $\pi(Z)$ closed. In part., π is surjective
2. For $Z_1, Z_2 \subseteq X$ " , $\pi(Z_1 \cap Z_2) = \pi(Z_1) \cap \pi(Z_2)$.
   In particular, $\pi(x) = \pi(y) \iff \overline{Gx} \cap \overline{Gy} \neq \emptyset$
3. π is universal for ~~maps~~ G-invariant maps to schemes

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "G\text{-inv}"] & \\ \operatorname{Spec} A^G \arrow[r, dashed, "\exists !"] & Y \text{ scheme} \end{tikzcd}$$

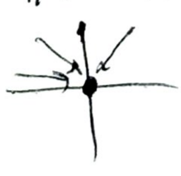
4. A fin. gen$/k \implies A^G$ fin gen$/k$ (Hilbert's 14th)

Cor: $X = \operatorname{Spec} A \to \operatorname{Spec} A^G = X/\!\!/G$ induces bijection between closed orbits and closed pts of $X/\!\!/G$.

$\uparrow$ GIT quotient or <u>good quotient</u>

Ex 1 $\mathbb{G}_m$ ~~Spec~~ $\curvearrowright \mathbb{A}^n$ via $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$

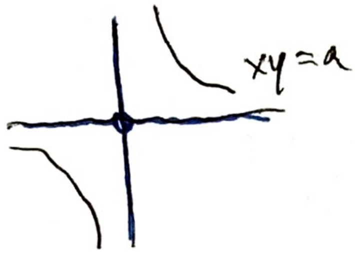
$$\begin{array}{c} \mathbb{A}^n \\ \downarrow \\ \mathbb{A}^n/\!\!/\mathbb{G}_m = \operatorname{Spec} k \end{array}$$



but $(\mathbb{A}^n \setminus 0)/\mathbb{G}_m = \mathbb{P}^{n-1}$

Ex 2 $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ via $t\cdot(x,y)=(tx, t^{-1}y)$

$$\begin{array}{c} \mathbb{A}^2 \\ \big\downarrow \pi \\ \mathbb{A}^2/\!/\mathbb{G}_m = \operatorname{Spec} k[xy] \end{array}$$



Here ~~$\mathbb{A}$~~ $\mathbb{G}_m \curvearrowright \mathbb{A}^2\setminus 0$ freely ~~&~~ w/ quotient $\mathbb{A}^1 \cup_{\mathbb{A}^1\setminus 0} \mathbb{A}^1$

Pf is <u>not</u> hard (we will give it later).

<u>Rmk</u>: Proof of (4) is much easier in this case.

Sketch of (4): G lin red $\Rightarrow \forall I \subset A^G \quad IA \cap A^G = I$ (later)

$\Rightarrow A^G$ is noeth.

Choose G-equiv. $k[x_1,\dots,x_n] \twoheadrightarrow A$ with kernel J

$\Rightarrow A^G = (k[x_1,\dots x_n]/J)^G = k[x_1,\dots,x_n]^G / J^G$

Suffices to show $k[x_1,\dots x_n]^G$ fin gen/k but this is graded (with k in deg 0) and noeth, so fin gen/k.

<u>Projective setup</u>: $X \subseteq \mathbb{P}(V)$ G-inv. closed subscheme w/ V fin. dim'l G-rep'n

($\mathcal{O}_X(1)$ is ample G-linearization, i.e. has G-action)

Define the <u>semistable locus</u> as

$$X^{ss} = \{x \in X \mid \exists f \in \Gamma(X, \mathcal{O}(d))^G,\ d>0 \text{ and } f(x)\neq 0\}$$

<u>Main Thm (proj case)</u>

$$\pi: X^{ss} \longrightarrow X^{ss}/\!/G = \operatorname{Proj} \underbrace{\bigoplus_{d\geq 0} \Gamma(X,\mathcal{O}(d))^G}_{R^G}$$ satisfies (1)-(3) and

$X^{ss}/\!/G$ is projective.

<u>Pf</u>: For $f \in \Gamma(X,\mathcal{O}(d))^G$, $d>0$

$$\begin{array}{ccccc} \operatorname{Spec} R_{(f)} & \subset & X^{ss} & \subseteq & X = \operatorname{Proj} R \\ \big\downarrow & & \big\downarrow & & \\ \operatorname{Spec} (R^G)_{(f)} & \subset & X^{ss}/\!/G = \operatorname{Proj} R^G & & \end{array}$$

& (1)-(3) are local

$(R^G)_{(f)} = (R_{(f)})^G$

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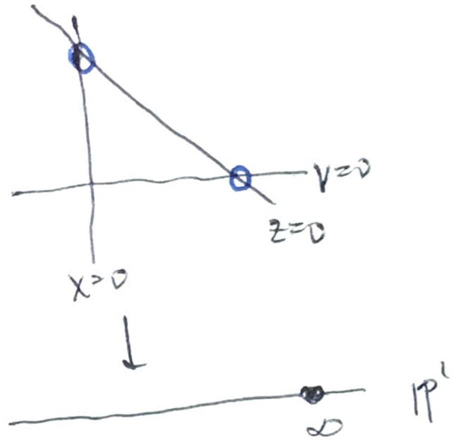
<u>Ex:</u> $\mathbb{G}_m \curvearrowright \mathbb{P}^2 \quad t\cdot[x:y:z] = [tx : t^{-1}y : z]$

$\rightsquigarrow k[x,y,z]^{\mathbb{G}_m} = k[xy, z]$

$$\begin{array}{ccl} [x:y:z] & (\mathbb{P}^2)^{ss} & = \mathbb{P}^2 \setminus V(xy, z) \\ \big\downarrow & \big\downarrow & \\ [xy:z] & (\mathbb{P}^2)^{ss} /\!/ \mathbb{G}_m & = \operatorname{Proj} k[xy, z] = \mathbb{P}^1 \end{array}$$



Good: get projective quotient for free

Bad: Difficult to determine X^{ss}, even to show non-empty

<u>Hilbert-Mumford criterion</u>

Let V be a G-repn & $0 \neq v \in V$

Then $v \in \mathbb{P}(V)$ not semistable $\iff 0 \in \overline{Gv}$

$\iff \exists\, \lambda : \mathbb{G}_m \to G,\ \lim_{t\to 0} \lambda(t)v = 0$

"$\Rightarrow$" is hard

(reduces determining X^{ss} to a combinatorial question)

§2. Algebraic stacks

Defn

① An alg. space is a sheaf X on $Sch_{\acute{e}t}$ s.t.

$\exists\ U \longrightarrow X$ repr. by schemes, étale & surjective.

(U scheme)

② An alg. stack is a stack $\mathcal{X}$ on $Sch_{\acute{e}t}$ s.t.

$\exists\ U \longrightarrow \mathcal{X}$ representable, smooth & surjective.

(presentation) (U scheme)

Rmk: ① sheaf: functor $Sch \to Sets$ s.t. objects glue

stack: "functor" $Sch \to Groupoids$ s.t. objects glue

② The diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is repn.

Prop ① If G finite group acting freely on a scheme U, U/G, defined as sheafification of $S \mapsto U(S)/G(S)$, is an alg. space

② If G is a smooth alg gp/k $\curvearrowright$ scheme/k U, then $[U/G]$, defined as stackification of $S \mapsto [U(S)/G(S)]$, is an alg. stack

Pf of ②: An objects of $[U/G](S)$ is data

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "G\text{-bdl}"] & U \text{ equiv} \\ S & \end{tikzcd}$$

and there is a cartesian diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "sm"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "\substack{sm\\ surj\\ repr}"] \\ S \arrow[r, "x"] & {[U/G]} \end{tikzcd}$$

Defn: $\mathcal{X}$ has affine diagonal $\iff$ $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is affine

Equiv. $\forall a, b \in \mathcal{X}(S)$ $Isom_S(a,b) \to S$ affine

$\Longrightarrow \forall x \in \mathcal{X}(k)$ $G_x = Isom_k(x,x)$ is affine

Ex: U affine diag (eg separated) $\Rightarrow$ $[U/G]$ affine diag

<u>Defn</u> Given an alg. stack $\mathcal{X}$, define site $\mathcal{X}_{lis\text{-}ét}$ where

objects: $\underset{\uparrow \text{scheme}}{U} \xrightarrow{sm} \mathcal{X}$

morphisms: étale maps $U \to U'$ over $\mathcal{X}$

covers: étale covers

We have a sheaf $\mathcal{O}_{\mathcal{X}} := \mathcal{O}_{\mathcal{X}_{lis\text{-}et}}$ where $\mathcal{O}_{\mathcal{X}}(U \xrightarrow{sm} \mathcal{X}) = \mathcal{O}_U(U)$

$\rightsquigarrow$ notion of $\mathcal{O}_{\mathcal{X}}$-module.

<u>Defn</u> We say a sheaf F of $\mathcal{O}_{\mathcal{X}}$-modules is <u>quasi-coherent</u> if

(1) $\forall \underset{\uparrow \text{scheme}}{U} \xrightarrow{sm} \mathcal{X}$ $\quad F|_{U_{zar}}$ q. coh. $\mathcal{O}_U$-mod.

(2) $\forall$
$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "sm"'] & & U' \arrow[dl, "sm"] \\ & \mathcal{X} & \end{tikzcd}$$
$f^*(F|_{U_{zar}}) \cong F|_{U'_{zar}}$

When $\mathcal{X}$ is <u>noetherian</u> (i.e. qc, qs & $\exists$ noeth presentation),

say F is <u>coherent</u> if F is q coh & $\forall U \xrightarrow{sm} \mathcal{X}$ $\quad F|_{U_{zar}}$ coh.

<u>Rmk</u>: Can think of F as

$(\underset{\uparrow \text{scheme}}{S} \to \mathcal{X}) \longmapsto$ q. coh F_S on S

<u>Ex</u>: 1 For $\mathcal{M}_g$, $(S \longrightarrow \mathcal{M}_g) \longmapsto F_S := \pi_* \Omega_{C/S}$

$$\updownarrow$$

$$\underset{\text{sm. family of curves}}{C \xrightarrow{sm\ \pi} S}$$

<u>Fact</u>: Given $f: \mathcal{X} \to \mathcal{Y}$ qcqs, $\exists$ adjoint functors

$$\begin{tikzcd} QCoh(\mathcal{X}) \arrow[r, shift left, "f_*"] & QCoh(\mathcal{Y}) \arrow[l, shift left, "f^*"] \end{tikzcd}$$

(Can work w/ alg. stacks & q.coh sheaves just like schemes...)

Ex:

$$\begin{tikzcd} & M \arrow[d, no head] & \\ \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] \arrow[d, "q"] & \operatorname{Spec} A^G \\ & BG & \end{tikzcd}$$

$QCoh([\operatorname{Spec} A/G]) = \{A\text{-mod w/ } G\text{-action}\}$

Ex: Write $BG = [\operatorname{Spec} k/G] \rightsquigarrow QCoh(BG) = Rep(G)$

Then for M,

$\hookrightarrow p^* M = M$ forgetting G-action

$\hookrightarrow q_* M = M$ forgetting A-mod structure

$\hookrightarrow \pi_* M = M^G$ G-invariant

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§3. Good moduli spaces

Defn: A qcqs morphism $\pi: \mathcal{X} \to X$ (alg. space) is a good mod space (gms) if

(1) $\pi_*: QCoh(\mathcal{X}) \to QCoh(X)$ exact

(2) $\mathcal{O}_X \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathcal{X}}$

Ex: $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is gms if G lin. red.

- Good: defn is remarkably simple & implies desirable geom propert.
- Bad: hard to verify unless $\mathcal{X} = [\operatorname{Spec} A/G]$

THM Let $\pi: \mathcal{X} \to X$ be a gms defined over base S.

① π univ. closed & surjective

② For $Z_1, Z_2 \subseteq \mathcal{X}$ closed, $\pi(Z_1 \cap Z_2) = \pi(Z_1) \cap \pi(Z_2)$.

In part, for $x, y \in \mathcal{X}(k)$ $\pi(x) = \pi(y) \iff \overline{\{x\}} \cap \overline{\{y\}} \neq \emptyset$ in $\mathcal{X} \times_S k$

③ π is universal for maps to alg. spaces

④ $\mathcal{X}$ f.type/S & S noeth $\Rightarrow$ X f.type/S & π_* preserve coherence