

LECTURE 2: Properties of good moduli spaces

Review: A stack \mathcal{X} over $Sch_{\mathbb{Z}}$ is an assignment

$$Sch \rightarrow \text{Groupoids}, \quad S \mapsto \mathcal{X}(S)$$

cat. of objects over S^*

s.t. objects and morphism glue.

An alg. stack \mathcal{X} ~~is a stack with affine diagonal~~ with affine diagonal is a stack \mathcal{X} s.t.

$$\exists U \xrightarrow{\quad} \mathcal{X} \text{ affine, smooth and surjective}$$

↑ sep. scheme ↑ called a presentation

Ex: Let C smooth projective ^{conn} curve / $k = \mathbb{C}$
 Fix $r, d \in \mathbb{Z}$.

Define

$$\mathcal{M} : Sch/k \rightarrow \text{Groupoids}$$

$$S \mapsto \left\{ \begin{array}{l} \text{vector bds } E \text{ on } C \times S \text{ s.t. } \forall s \in S \\ \text{the fiber } E_s \text{ on } C_{k(s)} \text{ has rank } r, \text{ degree } d \end{array} \right\}$$

where $\text{Mor}(E, E') = \left\{ \text{isom } E \xrightarrow{\sim} E' \right\}$

Prop Last time: C smooth alg gp/k $\Rightarrow U$ sep schem/k
 $\Rightarrow [U/k]$ alg. stack w/ affine diag.

Prop: \mathcal{M} is an alg. stack with affine diag.

Pf Recall if E has rk r & deg d

(1) Riemann-Roch \Rightarrow Hilbert poly $P(n) = \chi(E(n)) = \text{deg } E(n) + \text{rk}(E(n))(1-g)$
 $= d + rn + r(1-g)$

(2) $\exists m \quad P(C, E(m)) \otimes \mathcal{O}_C \rightarrow E(m)$ & matrix (\Rightarrow induces iso on H^0)
 $\Rightarrow E \in \text{Quot}_C^P(\mathcal{O}_C(-m)^{\oplus P(m)})$ but depends on choice of basis

Let $Q_m \subseteq \text{Quot}_{\mathbb{C}}^P(E(m)^{P(m)})$ be subscheme of quotients
inducing iso on H^0 .

Let $M_m \subseteq M$ vect. bds E s.t. $E(m)$ glbs. gen.

Then $M_m = [Q_m / GL_{P(m)}]$ alg. stack w/ affine diag.

$$\leadsto M = \bigcup M_m$$

Recall from Lecture 1

↳ define good mod. space

↳ Main thm

Remark: Generalizes affine case of \mathcal{L}, Π

Lemma 1 If $\pi: X \rightarrow X$ is a gms and $F \in \mathcal{QCoh}(X)$,

~~$\pi^* \pi_* F \cong F$~~ $\pi^* F \cong \pi_* \pi^* F$

Pf: Étale local \Rightarrow Can assume $X = \text{Spec } A$.

Choose
$$\begin{array}{ccccc} \mathcal{O}_X^J & \rightarrow & \mathcal{O}_X^I & \rightarrow & F \rightarrow 0 \text{ free pres} \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathcal{O}_X^J & \rightarrow & \mathcal{O}_X^I & \rightarrow & \pi_* \pi^* F \rightarrow 0 \end{array}$$

π^* right exact & π_* exact $\Rightarrow \pi^* \pi_*$ right exact.

Lemma 2 Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \text{ } \mathbb{A}^1\text{-alg stack} \\ \downarrow \pi' & \square & \downarrow \pi \\ X' & \xrightarrow{g} & X \text{ } \mathbb{A}^1\text{-spaces} \end{array}$$

- 1) π gms $\Rightarrow \pi'$ gms
- 2) g fppl & π' gms $\Rightarrow \pi$ gms

Pf: (2) flat base change $\Rightarrow g^* \pi_* \cong \pi'_* g'^*$ $\Rightarrow \pi_*$ exact

\Downarrow faithfully exact $\Rightarrow \mathcal{O}_X \rightarrow \pi_* \mathcal{O}_X$ is iso (since its pullback via g is iso)

(1) By (2), can assume $X = \text{Spec } A$ & $X' = \text{Spec } A'$

$$g_* \pi'_* \cong \pi'_* g'_* \Rightarrow \pi'_* \text{ exact}$$

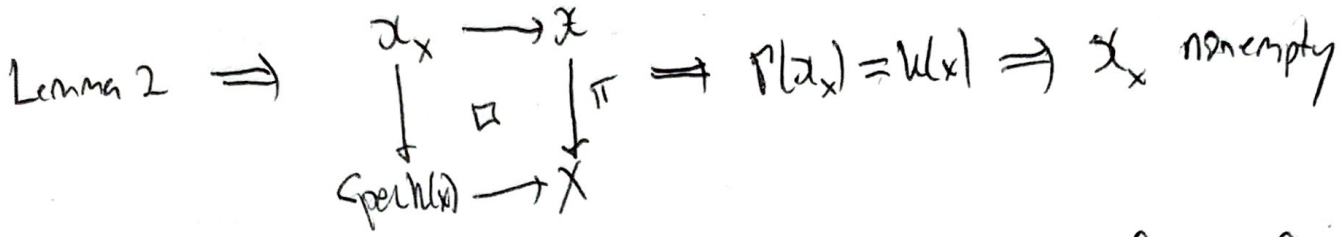
Lemma 1 $\Rightarrow g_* \mathcal{O}_{X'} \cong \pi_* \pi'_* g'_* \mathcal{O}_{X'} \cong \pi_* g'_* \mathcal{O}_{X'} \cong g_* \pi'_* \mathcal{O}_{X'}$

$$\Rightarrow \mathcal{O}_{X'} \cong \pi'_* \mathcal{O}_{X'} \quad \text{since } g \text{ affine}$$

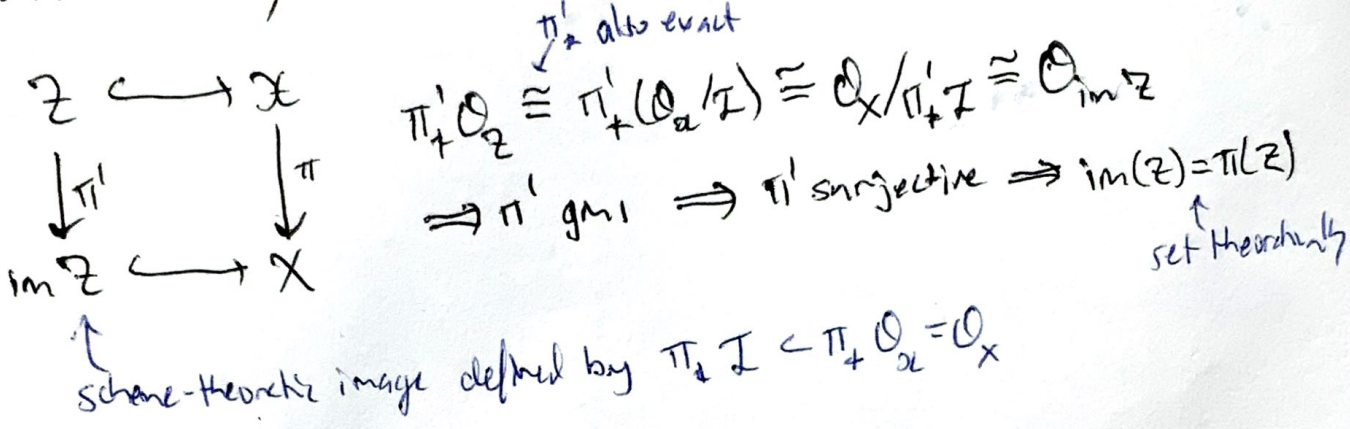
$$\begin{array}{ccc} X = \text{Spec } A & \xrightarrow{g'} & X \\ \downarrow & & \downarrow \pi \\ X' = \text{Spec } A' & \xrightarrow{g} & X' \end{array}$$

PF OF THM

① For surjectivity, let $x \in X$

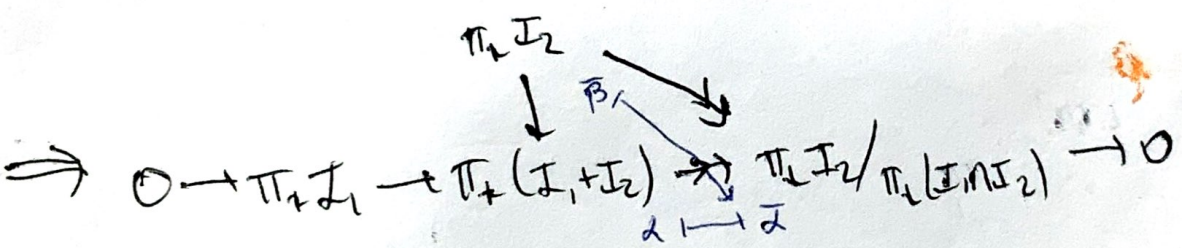


For univ. closedness, let $Z \subseteq \mathcal{X}$ defined by $I \subset \mathcal{O}_{\mathcal{X}, Z} \rightarrow \mathcal{O}_Z$



② For $Z_1, Z_2 \subset \mathcal{X}$ defined by I_1, I_2

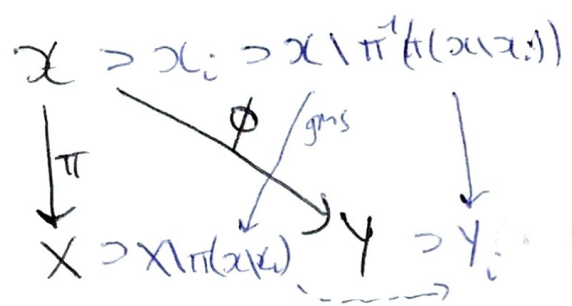
$$0 \rightarrow I_1 \rightarrow I_1 + I_2 \rightarrow I_2 / I_1 \cap I_2 \rightarrow 0$$



$$\Rightarrow \pi_+ I_1 + \pi_+ I_2 \xrightarrow{\sim} \pi_+(I_1 + I_2)$$

Reason: Locally element $\alpha \in \pi_+(I_1 + I_2) \mapsto \bar{\alpha}$
 lift to $\beta \Rightarrow \alpha - \beta \mapsto 0 \Rightarrow \beta$ is in image of $\pi_+ I_1$

(3)



Cases:

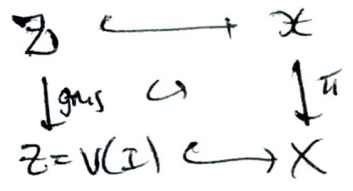
- ↳ Y affine: $\text{Mor}(\alpha, Y) = \text{Hom}(F(Y), F(\alpha)) = \text{Hom}(F(Y), F(X)) = \text{Hom}_{\text{Mor}(X, Y)}$
- ↳ Y scheme: $Y = \cup Y_i$ affine cover
- ↳ Y alg. space: harder

ideal sheaf of $\pi^{-1}(V(I))$

(4) First step: X noeth.

Suffices to show that if $I \subset \mathcal{O}_X$ ideal, $I = \pi_+ (\pi^{-1} I \cdot \mathcal{O}_X)$

$$0 \rightarrow I \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$



$$0 \rightarrow \pi^* I \rightarrow \pi^* \mathcal{O}_X \rightarrow \pi^* \mathcal{O}_Z \rightarrow 0$$

$$0 \rightarrow \pi_+ (\pi^* I \cdot \mathcal{O}_X) \rightarrow \pi_+ \mathcal{O}_X \rightarrow \pi_+ \mathcal{O}_Z \rightarrow 0$$

$\begin{matrix} \cong & & \cong \\ \mathcal{O}_X & & \mathcal{O}_Z \end{matrix}$

Need algebra result:

(or $X \rightarrow Y$ univ. submersive)

