LECTURE 2: Properties of good moduli spaces

Review: ↳ A stack $\mathcal{X}$ over Sch_{et} is an assignment

$$Sch \longrightarrow \text{Groupoids}, \quad S \longmapsto \underbrace{\mathcal{X}(S)}_{\text{cat. of "objects over } S\text{"}}$$

s.t. objects and morphism glue.

An alg. stack with affine diagonal is a stack $\mathcal{X}$ s.t.

$$\exists\ \underbrace{U \longrightarrow \mathcal{X}} \text{ affine, smooth and surjective}$$

↑ sep. scheme ↑ called a presentation

Ex: Let C smooth proj. conn curve / $k = \bar{k}$

Fix $r, d \in \mathbb{Z}$.

Define

$$\mathcal{M} : Sch/k \longrightarrow \text{Groupoids}$$

$$S \longmapsto \left\{ \begin{array}{l} \text{vector bdls } E \text{ on } C\times S \text{ s.t. } \forall s \in S \\ \text{the fiber } E_s \text{ on } C_{k(s)} \text{ has rank } r, \text{ degree } d \end{array} \right\}$$

where $Mor(E, E') = \{\text{isom } E \xrightarrow{\sim} E'\}$

~~Prop~~ Last time: G smooth alg gp/k $\curvearrowright$ U sep schem/k

$\Rightarrow$ $[U/G]$ alg. stack w/ affine diag.

Prop: $\mathcal{M}$ is an alg. stack with affine diag.

Pf Recall if E has rk r & deg d

(1) Riemann-Roch $\Rightarrow$ Hilbert poly $P(n) = \chi(E(n)) = \deg E(n) + rk(E(n))(1-g)$
$= d + rn + r(1-g)$

(2) $\exists m$ $\Gamma(C, E(m)) \otimes \mathcal{O}_C \twoheadrightarrow E(m)$ & (⇒ induces iso on H^0)

$\Rightarrow E \in Quot^P_C(\mathcal{O}_C(-m)^{\oplus P(m)})$ but depends on choice of basis

Let $Q_m \subseteq \mathrm{Quot}^P_C(E(-m)^{P(m)})$ be subscheme of quotients inducing iso on H^0.

Let $\mathcal{M}_m \subseteq \mathcal{M}$ vect. bdls E s.t. $E(m)$ glob. gen.

Then $\mathcal{M}_m = [Q_m / GL_{P(m)}]$ alg. stack w/ affine diag.

$\rightsquigarrow \mathcal{M} = \bigcup \mathcal{M}_m$

Recall from Lecture 1

↳ defn good mod. space

↳ Main thm

Rmk: Generalizes affine case of GIT ③

Lemma 1 If $\pi : \mathcal{X} \to X$ is a gms and $F \in QCoh(X)$,

~~π_*~~ $F \xrightarrow{\sim} \pi_* \pi^* F$

Pf: Étale local $\Rightarrow$ Can assume $X = \operatorname{Spec} A$.

Choose

$$\begin{tikzcd}
\mathcal{O}_X^J \arrow[r] \arrow[d, "\wr"] & \mathcal{O}_X^I \arrow[r] \arrow[d, "\wr"] & F \arrow[r] \arrow[d, "\wr"] & 0 \\
\mathcal{O}_X^J \arrow[r] & \mathcal{O}_X^I \arrow[r] & \pi_* \pi^* F \arrow[r] & 0
\end{tikzcd}$$

free pres

π^* right exact & π_* exact $\Rightarrow$ $\pi^* \pi_*$ right exact. ⊠

Lemma 2 Consider a cartesian diagram

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\
X' \arrow[r, "g"] & X
\end{tikzcd}$$

$\leftarrow$ alg stack

$\leftarrow$ alg. spaces

1) π gms $\Rightarrow$ π' gms

2) g fppf & π' gms $\Rightarrow$ π gms

Pf: (2) flat base change $\Rightarrow$ $g^* \pi_* \simeq \pi'_* g'^*$ $\Rightarrow$ π_* exact

$\uparrow$ faithfully exact (under g^*) $\quad \uparrow\uparrow$ exact (under $\pi'_* g'^*$)

$\Downarrow$

$\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is iso (since its pullback via g is iso)

(1) By (2), can assume $X = \operatorname{Spec} A$ & $X' = \operatorname{Spec} A'$

$g_* \pi'_* \simeq \pi_* g'_*$ $\Rightarrow$ π'_* exact

$\uparrow$ faithfully exact (under g_*) $\quad \uparrow\uparrow$ exact (under $\pi_* g'_*$)

Lemma 1 $\Rightarrow$ $g_* \mathcal{O}_{X'} \xrightarrow{\sim} \pi_* \pi^* g_* \mathcal{O}_{X'} \simeq \pi_* g'_* \mathcal{O}_{\mathcal{X}'} \simeq g_* \pi'_* \mathcal{O}_{\mathcal{X}'}$

$\uparrow$ since g affine

$\Rightarrow$ $\mathcal{O}_{X'} \xrightarrow{\sim} \pi'_* \mathcal{O}_{\mathcal{X}'}$

$$\begin{tikzcd}
\mathcal{X}' = \operatorname{Spec}_{\mathcal{X}} \pi^* g_* \mathcal{O}_{X'} \arrow[r, "g'"] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\
X' = \operatorname{Spec}_X g_* \mathcal{O}_{X'} \arrow[r, "g"] & X
\end{tikzcd}$$

PF OF THM

① For surjectivity, let $x \in X$

Lemma 2 $\Rightarrow$
$$\begin{tikzcd} \mathcal{X}_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}\, k(x) \arrow[r] & X \end{tikzcd}$$
$\Rightarrow \Gamma(\mathcal{X}_x) = k(x) \Rightarrow \mathcal{X}_x$ nonempty

For univ. closedness, let $Z \subseteq \mathcal{X}$ defined by $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}} \twoheadrightarrow \mathcal{O}_Z$

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\pi'"] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{im}\, Z \arrow[r, hook] & X \end{tikzcd}$$

π'_* also exact

$\pi'_* \mathcal{O}_Z \cong \pi'_*(\mathcal{O}_{\mathcal{X}}/\mathcal{I}) \cong \mathcal{O}_X/\pi'_*\mathcal{I} \cong \mathcal{O}_{\mathrm{im}\, Z}$

$\Rightarrow \pi'$ gm1 $\Rightarrow \pi'$ surjective $\Rightarrow \mathrm{im}(Z) = \pi(Z)$

set theoretically

scheme-theoretic image defined by $\pi_* \mathcal{I} \subset \pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$

② For $Z_1, Z_2 \subset \mathcal{X}$ defined by $\mathcal{I}_1, \mathcal{I}_2$

$$0 \to \mathcal{I}_1 \to \mathcal{I}_1 + \mathcal{I}_2 \to \mathcal{I}_2/\mathcal{I}_1 \cap \mathcal{I}_2 \to 0$$

$$\begin{tikzcd} & & \pi_*\mathcal{I}_2 \arrow[d] \arrow[dr] \arrow[dr, "\bar{\beta}"', bend right] & & \\ 0 \arrow[r] & \pi_*\mathcal{I}_1 \arrow[r] & \pi_*(\mathcal{I}_1 + \mathcal{I}_2) \arrow[r] & \pi_*\mathcal{I}_2/\pi_*(\mathcal{I}_1 \cap \mathcal{I}_2) \arrow[r] & 0 \end{tikzcd}$$
$\Rightarrow$

$\alpha \longmapsto \bar{\alpha}$

$\Rightarrow \pi_*\mathcal{I}_1 + \pi_*\mathcal{I}_2 \xhookrightarrow{\sim} \pi_*(\mathcal{I}_1 + \mathcal{I}_2)$

Reason: Locally element $\alpha \in \pi_*(\mathcal{I}_1 + \mathcal{I}_2) \longmapsto \bar{\alpha}$
Lift to $\bar{\beta} \Rightarrow \alpha - \bar{\beta} \longmapsto 0 \Rightarrow$ α is in image of $\pi_*\mathcal{I}_1$

③

③

$$\begin{tikzcd}
\mathcal{X} \arrow[d, "\pi"] \arrow[drr, "\phi"] & \supset \mathcal{X}_i & \supset \mathcal{X}\setminus\pi^{-1}\pi(\mathcal{X}\setminus\mathcal{X}_i) \arrow[dl, "gms"] \arrow[dr] & \\
X & \supset X\setminus\pi(\mathcal{X}\setminus\mathcal{X}_i) \arrow[rr, dashed, bend right] & Y & \supset Y_i
\end{tikzcd}$$

Cases:

↳ Y affine: $\mathrm{Mor}(\mathcal{X},Y) = \mathrm{Hom}(\Gamma(Y),\Gamma(\mathcal{X})) = \mathrm{Hom}(\Gamma(Y),\Gamma(X)) =$ ~~Hom~~ $\mathrm{Mor}(X,Y)$

↳ Y scheme: $Y = \bigcup Y_i$ affine cover

↳ Y alg. space: harder

④ First step: X noeth.

ideal sheaf of $\pi^{-1}(V(I))$

Suffices to show that if $I \subset \mathcal{O}_X$ ideal, $I = \pi_*(\pi^{-1}I \cdot \mathcal{O}_{\mathcal{X}})$

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, hook] \arrow[d, "gms"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{X} \arrow[d, "\pi"] \\
Z = V(I) \arrow[r, hook] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\pi^{-1}I \arrow[dr] & & & & \\
0 \arrow[r] & \pi^{-1}I\cdot\mathcal{O}_{\mathcal{X}} \arrow[r] & \mathcal{O}_{\mathcal{X}} \arrow[r] & \mathcal{O}_{\mathcal{Z}} \arrow[r] & 0 \\
0 \arrow[r] & \pi_*(\pi^{-1}I\cdot\mathcal{O}_{\mathcal{X}}) \arrow[r] & \pi_*\mathcal{O}_{\mathcal{X}} \arrow[r] \arrow[d, phantom, "\parallel"] & \pi_*\mathcal{O}_{\mathcal{Z}} \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
& & \mathcal{O}_X & \mathcal{O}_Z &
\end{tikzcd}$$

✓

Need algebra result:

(or $X \to Y$ univ. submersive)

$$\begin{tikzcd}
U \arrow[d, "\substack{qc\\ \text{univ. subm.}}"'] \arrow[d, "\substack{\text{~~qa~~}\\ \text{~~univ. closed~~}}"] \arrow[dr, bend left, "f.type"] & \\
X \arrow[r] & S
\end{tikzcd}$$

X noeth $\quad S$ noeth $\quad \boxed{\Rightarrow \text{f.type}}$

~~univ. closed~~

$\rightsquigarrow$ Apply to

$$\begin{tikzcd}
U \arrow[r, "sm"] \arrow[dr, "\text{univ. subm.}"'] & \mathcal{X} \arrow[d] \arrow[dr] & \\
& X \arrow[r] & S
\end{tikzcd}$$