LECTURE 3

§2. Tannaka duality & applications

Ques: What invariants of a scheme are sufficient to recover it?

Gabriel ('62): For qs schemes $X \cong Y \iff QCoh(X) \cong QCoh(Y)$ (as abstract cats)

(Gabriel proved noeth case and Rosenberg generalized)

Other exs: ↳ $(Rep(G), \otimes)$ ⇝ G for alg. gp G

↳ Ringed topoi $(Sh(X_{et}), \mathcal{O}_X)$ ⇝ X

↳ $(Perf(X), \otimes)$ ⇝ X (Balmer)

↳ If X geom irred, normal, proj of $\dim X \geq 4$, $|X|$ Zariski top. space recovers X (KLOS)

What about morphisms?

Thm (Lurie, Hall-Rydh, Bhatt-Halpern-Leistner)

Let $\mathcal{X}$ & $\mathcal{Y}$ be noeth alg stacks w/ affine diagonal

Then $Mor(\mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \underbrace{Mor_{\otimes}(Coh(\mathcal{Y}), Coh(\mathcal{X}))}_{\text{functors } F: Coh(\mathcal{Y}) \to Coh(\mathcal{X}) \text{ right exact \& preserve } \otimes}$, $f \mapsto f^*$

Proof (We will only show) that

Essential surjectivity: Let $F: Coh(\mathcal{Y}) \to Coh(\mathcal{X})$

Preliminaries:

↳ Can assume $\mathcal{X} = \operatorname{Spec} A$:
$$\begin{tikzcd} \operatorname{Spec} B \arrow[d, shift left] \arrow[d, shift right] & \\ \operatorname{Spec} A \arrow[d, "g"', "sm"] \arrow[dr, dashed, "g^* \circ F"] & \\ \mathcal{X} \arrow[r, dashed, "F"] & \mathcal{Y} \end{tikzcd}$$

↳ F extends to a $F: QCoh(\mathcal{Y}) \to QCoh(\mathcal{X})$ $\otimes$-functor preserving colimits

Reason: Any q-coh sheaf is a filtered limit of coherent sheaves

MAIN IDEA

Step 1 True if $\mathcal{Y} = \operatorname{Spec} B$

Step 2 Reduce to affine case by choosing $V = \operatorname{Spec} B \twoheadrightarrow \mathcal{Y}$

$$\begin{tikzcd}
U = \operatorname{Spec} C \arrow[r, dashed, "F'"] \arrow[d] & V = \operatorname{Spec} B \arrow[d, "g \text{ affine}"] \\
\mathcal{X} = \operatorname{Spec} A \arrow[r, dashed, "F"] & \mathcal{Y}
\end{tikzcd}$$

Define $C = F(g_* \mathcal{O}_V)$ A-alg. (since $g_* \mathcal{O}_V$ is $\mathcal{O}_{\mathcal{Y}}$-alg)

$F' : \operatorname{Mod}(B) \to \operatorname{Mod}(C)$ $\otimes$-functor preserving colimits

$$M \longmapsto F(\underbrace{g_* \tilde{M}}_{g_* \mathcal{O}_V\text{-mod}})$$

mod / $C = F(g_* \mathcal{O}_V)$

Need to show C f. flat A-alg.

Then: $F' = (f')^*$ for $f' : \operatorname{Spec} C \to \operatorname{Spec} B$

and f' descend to $f : \mathcal{X} \to \mathcal{Y}$.

Details

① $B \xrightarrow{\sim} \operatorname{End}(\mathcal{O}_V) \xrightarrow{F'} \operatorname{End}(\mathcal{O}_U) \cong A$ ring map

② (hard step) Goal: $A \to C$ f. flat

Claim: $A \to C$ universally injective.

Defn • A map $M \to N$ of A-mod is univ. inj if $\forall P \in \operatorname{Mod}(A)$ $M \otimes_A P \hookrightarrow N \otimes_A P$

• A ring map $A \to B$ is univ inj if it is as a map in A-mod

Properties

① $A \to B$ f. flat $\Rightarrow$ $A\to B$ univ. inj

② $M \to N$ split injective $\Rightarrow$ univ. injective.

The converse holds if N is fin. pres.

③ If $A \to A'$ f. flat, then $M \to N$ univ inj $\Leftrightarrow$ $M \otimes_A A' \to N \otimes_A A'$ univ. inj

④ If $A \to C$ univ inj, then $A \to C$ f. flat $\Leftrightarrow$ $C \to C \otimes_A C$ f. flat

$c \mapsto c \otimes 1$

(Aside) ⑤ $A \to B$ univ. inj $\Leftrightarrow$ $A \to B$ satisfies eff. descent for modules

$\left(\text{i.e. } \mathrm{Mod}(A) = \{ N \in \mathrm{Mod}(B),\ \alpha : N \otimes_{B,p_1} (B \otimes_A B) \xrightarrow{\sim} N \otimes_{B,p_2} (B \otimes_A B) \text{ s.t. } \alpha \text{ satisfies cocycle} \}\right)$

Recall that we have

$$\begin{tikzcd}
\mathrm{Spec}\, C \otimes_A C \arrow[r, "f''"] \arrow[d, shift left] \arrow[d, shift right, "\text{f. flat}"'] & V \times_Y V \text{ affine} \arrow[d, shift left, "\text{f. flat}"] \arrow[d, shift right] \\
\mathrm{Spec}\, C \arrow[r, "f'"] \arrow[d] & V = \mathrm{Spec}\, B \arrow[d, "\substack{\text{smooth} \\ \text{affine}} \Rightarrow \text{f. flat}"] \\
\mathrm{Spec}\, A = X \arrow[r, dashed, "F"] & Y
\end{tikzcd}$$

so that univ. inj is well-defined

↳ $\mathcal{O}_Y \to g_* \mathcal{O}_V$ univ. inj (by ① and ③)

↳ Write $g_* \mathcal{O}_V = \mathrm{colim}\, G_i$ for $G_i \subset g_* \mathcal{O}_V$ coherent

↳ $\mathcal{O}_Y \to G_i$ univ inj $\xRightarrow{②}$ split injective smooth-locally

↳ Apply F: $A = F(\mathcal{O}_Y) \to F(g_* \mathcal{O}_V) = C = \mathrm{colim}\, F(G_i)$

$A \to F(G_i)$ coherent, $F(G_i) \to \mathrm{colim}\, F(G_i)$

Need to show: $A \to F(G_i)$ split injective

↳ Have $G_i^\vee \twoheadrightarrow \mathcal{O}_Y^\vee = \mathcal{O}_Y$ (as $\mathcal{O}_Y \to G_i$ smooth-loc. split)

↳ So $F(G_i^\vee) \twoheadrightarrow F(\mathcal{O}_Y) = A$ (by right exactness)

$\exists\, \lambda \mapsto 1$

↳ The natural map $F(G_i^\vee) \to F(G_i)^\vee$ sends $\lambda \mapsto (F(G_i) \to A)$

section of $A \to F(G_i)$

Conclusion:

$A \to F(4_i)$ univ. inj $\Rightarrow A \to C = \operatorname{colim} F(4_i)$ univ. inj

$\Rightarrow A \to C$ f. flat by ④

By uniqueness in the affine case, $f' : \operatorname{Spec} C \to \operatorname{Spec} B$ descends to our desired map $f : \operatorname{Spec} A = \mathcal{X} \to \mathcal{Y}$

Applications

Defn A noeth alg stack $\mathcal{X}$ is coherently complete along ~~the~~ a closed substack $\mathcal{X}_0$ if

$$\operatorname{Coh}(\mathcal{X}) \xrightarrow{\sim} \varprojlim \operatorname{Coh}(\mathcal{X}_n)$$

nth nilpotent thickening

Cor Let $\mathcal{X}, \mathcal{Y}$ noeth alg. stacks w/ affine diag

Suppose $\mathcal{X}$ is coh. complete along $\mathcal{X}_0$

Then $\operatorname{Mor}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \varprojlim \operatorname{Mor}(\mathcal{X}_n, \mathcal{Y})$

Pf

$$\begin{aligned}
\operatorname{Mor}(\mathcal{X}, \mathcal{Y}) &\overset{TD}{\cong} \operatorname{Mor}_{\otimes}(\operatorname{Coh}(\mathcal{Y}), \operatorname{Coh}(\mathcal{X})) \\
&\overset{CC}{\cong} \operatorname{Mor}_{\otimes}(\operatorname{Coh}(\mathcal{Y}), \varprojlim \operatorname{Coh}(\mathcal{X}_n)) \\
&\cong \varprojlim \operatorname{Mor}_{\otimes}(\operatorname{Coh}(\mathcal{Y}), \operatorname{Coh}(\mathcal{X}_n)) \\
&\overset{TD}{\cong} \varprojlim \operatorname{Mor}(\mathcal{X}_n, \mathcal{Y})
\end{aligned}$$

■

Ex 1 R noeth I-adically complete (i.e. $R = \varprojlim R/I^n$)

$\Rightarrow \operatorname{Spec} R$ coh. complete along $\operatorname{Spec} R/I$

$\Rightarrow$ Compatible maps $\operatorname{Spec} R/I^n \to Y$ extends to $\operatorname{Spec} R \to Y$

sep scheme

(obvious when R is local)

Ex2 (Formal GAGA)

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d] & X \arrow[d, "\text{proper}"] \\
\operatorname{Spec} k \arrow[r, hook, "o"] & \operatorname{Spec} R
\end{tikzcd}$$

coh. complete

complete local noeth ring

<u>Thm</u> Let G lin reductive alg gp$/k=\bar{k}$
Let $X = \operatorname{Spec} A$ noeth. k-scheme w/ G-action.
Assume A^G complete local
Let $Gx \subseteq X$ unique closed orbit
Then $[X/G]$ coh complete along $[Gx/G] \cong BG_x$.

<u>Ex</u>: $[\mathbb{A}^n/\mathbb{G}_m] \supseteq B\mathbb{G}_m$

<u>Thm</u> (–Hall-Rydh) Let $\mathcal{X}$ be an alg. stack of f.type$/k=\bar{k}$ w/ affine diag
Let $x \in \mathcal{X}(k)$ with lin. red stab G_x
Then $\exists\ [\operatorname{Spec} A/G_x] \xrightarrow{\text{ét}} \mathcal{X}$ ngbd of x.

<u>Pf</u>: (Assuming $\mathcal{X}$ smooth at x)

$$\begin{tikzcd}
 & & & \mathcal{X} \ni x \\
BG_x \arrow[r, hook] & \mathcal{X}_1 \arrow[r, hook] & \cdots & \mathcal{X} \\
BG_x \arrow[r, hook] \arrow[u, "\wr"] & \tau_1 \arrow[r, hook] \arrow[u, dashed, "\wr"] & \cdots \arrow[r] & \hat{\tau} \arrow[r] \arrow[d] \arrow[u, dashed] \arrow[dr, phantom, "\square"] & \tau = [T_x/G_x] \ni 0 \arrow[d, "\pi"] \\
 & & & \operatorname{Spec} \hat{\mathcal{O}}_{T,\pi(0)} \arrow[r] & T = T_x /\!/ G_x
\end{tikzcd}$$

Key ingredient 1: defn theory (uses G_x lin red)

Zariski tangent space

Key ingredient 2: $\hat{\tau}$ is coh. complete along BG_x

<u>Final step</u>: Artin approximation

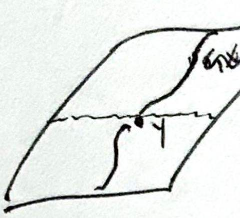
Let $F: \mathrm{Sch}/T \to \mathrm{Sets}$, $(T' \to T) \mapsto \operatorname{Mor}_T(\tau \times_T T', \mathcal{X})$ limit pres
Have $\hat{\xi} \in F(\hat{\mathcal{O}}_{T,\pi(0)}) \Rightarrow \exists\ (T', t') \to (T, 0)$ & $\xi' \in F(T')$ s.t.
$\hat{\xi}|_{\mathcal{O}_{T,\pi(0)}/m^2} \simeq \xi'|_{\mathcal{O}_{T,\pi(0)}/m^2} \Rightarrow \tau' \to \mathcal{X}$ étale at unique closed point over t'

Hilbert-Mumford criterion

G reductive$/k=\bar{k} \curvearrowright X = \operatorname{Spec} A$ f.t.type$/k$

For $x \in X(k)$, $\exists\, \lambda : \mathbb{G}_m \to G$ s.t. $\lim_{t\to 0} \lambda(t)x$ is in the unique closed orbit in $\overline{Gx}$.

Pf



Let $y \in \overline{Gx}$ have closed orbit

Choose curve

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "g"] \arrow[d] & G \arrow[r] & \overline{Gx} \arrow[d] \\ \operatorname{Spec} R \arrow[rr] & & X \end{tikzcd} \qquad h \mapsto hx, \quad 0 \mapsto y$$

$R = k[[\pi]]$ DVR

Suffices to show

Iwahori decomp: $\forall\, g \in G(K)\ \exists\, h_1, h_2 \in G(R)$ & $\lambda : \mathbb{G}_m \to G$

s.t. $g = h_1 \lambda|_K h_2 \in G(K)$.

$g \in G(K)$ determines $\operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \to BG$ by gluing trivial G-torsors w/ g.

$$\begin{tikzcd} & \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \arrow[r] \arrow[d] & BG \\ {[\mathbb{A}^2/_{1,-1}\mathbb{G}_m]} \arrow[d] \arrow[dr, phantom, "\square"] & {[\operatorname{Spec}(R[x,y]/(xy-\pi))/\mathbb{G}_m]} \arrow[l] \arrow[d, "gms"] \arrow[ur, dashed, "\exists"] & \\ \mathbb{A}^1 & \operatorname{Spec} R \arrow[l] & \end{tikzcd}$$

$\exists$ - exists b/c G-reductive (next time)

$t \cdot (x,y) = (tx, t^{-1}y)$

$$\begin{tikzcd} \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \arrow[r, "g \in G(K)"] \arrow[d] & BG \\ \mathcal{X} = [\operatorname{Spec} R[x,y]/(xy-\pi)/\mathbb{G}_m] \arrow[ur, "\tilde{g}"] \arrow[ur, shift right, "\lambda\circ\pi"'] \arrow[r, shift left, "\pi"] & B\mathbb{G}_m = \mathcal{X}_0 \arrow[l, shift left, "\circ i"] \arrow[u, "\lambda"'] \end{tikzcd}$$

gives $\lambda : \mathbb{G}_m \to G$

i.e. surjective on morphisms

$$\operatorname{Map}(\mathcal{X}, BG) \overset{cc}{\simeq} \lim \operatorname{Map}(\mathcal{X}_n, BG) \xrightarrow{\text{full}} \operatorname{Map}(B\mathbb{G}_m, BG)$$

$\tilde{g}, \lambda\circ\pi$

$\Uparrow$ def'n theory

Conclusion $\tilde{g} \simeq \lambda \circ \pi$

$\Longrightarrow$ g and $\lambda|_K \in G(K)$ define equivalent maps

$$\operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \longrightarrow BG$$

$\Longrightarrow$ $g = h_1 \lambda|_K h_2$ for some $h_1, h_2 \in G(R)$