Lecture 4: Construction of projective moduli

↳ The classical ~~old~~ construction uses GIT & Hilbert-Mumford crit

↳ We will pursue an intrinsic approach.

Setup: Let $\mathcal{M}^{ss}$ be the moduli problem of objects you wish to parametrize

Let $\mathcal{M}^{ss} \subset \mathcal{M}$ be a natural enlargement

Examples: ① $\mathcal{M}^{ss} = \overline{\mathcal{M}}_g \subset \mathcal{M} =$ moduli of all (poss. singular) curves

② $\mathcal{M}^{ss} =$ moduli of semistable vect. bdls on a fixed sm. curve C

with fixed rank & deg

$\cap$ (E is semistable $\Leftrightarrow \forall F \subsetneq E \quad \frac{\mathrm{rk}\, F}{\deg F} \leq \frac{\mathrm{rk}\, E}{\deg E}$)

$\mathrm{Bun}(C)_{r,d} \to \mathcal{M} =$ moduli of all vect. bdls or even coherent sheaves

③ $\mathcal{M}^{ss} =$ moduli of semistable objects w.r.t. a fixed Bridgeland stability cond on a sm proj var. X

$\cap$

$\mathcal{M}$ = moduli of all complexes on X or all objects in heart

④ $\mathcal{M}^{ss} =$ moduli of K-semistable Fanos

$\cap$

$\mathcal{M} =$ moduli of all Fanos.

The 6 steps toward projective moduli

① **Algebraicity**: show $\mathcal{M}$ is an alg. stack loc. of fin type/k

Last lecture: $\mathrm{Bun}(C)_{r,d}$ ✓

② **Openness of semistability** $\mathcal{M}^{ss} \subset \mathcal{M}$ is an open substack

③ **Boundedness of semistability** $\mathcal{M}^{ss}$ is of fin.type/k (i.e. quasi-cpt)

(Uses Hilb or Quot)

④ Existence of a moduli space $\mathcal{M}^{ss} \to M^{ss}$ ← alg. space

Trichotomy

No auts: already an alg. space ✓

Finite auts: Apply Keel-Mori Thm: $\mathcal{X} \xrightarrow{\pi} X$

If $\mathcal{X}$ is a separated DM stack, ∃ coarse mod. space with ~~X~~ sep. alg space

means ① bijective on k-pts

② univ. for maps to alg spaces

3rd case: infinite auts (but the closed (or "polystable") objects have reductive auts. ③

Apply A-HL-H: If $\mathcal{M}^{ss}$ $\mathcal{X}$ is an alg. stack of f.type / $k=\bar{k}$ (char 0) with affine diag such that

① $\mathcal{X}$ is Θ-reductive

② $\mathcal{X}$ is S-complete

Then $\exists \pi: \mathcal{X} \to X$ gms w/ X separated alg. space.

⑤ Semistable reduction: Show $\mathcal{M}^{ss}$ is universally closed via the existence part of the valuative criterion.

$\Longrightarrow$ the coarse/good mod. space M^{ss} is proper.

($\overline{\mathcal{M}}_g$: Deligne-Mumford & $Bun(C)^{ss}_{r,d}$ Langton's thm

⑥ Projectivity: Show that some tautological line bdl on $\mathcal{M}^{ss}$ descends to an ample line bdl on M^{ss}

For $\overline{\mathcal{M}}_g$, use Kollár's criterion

For $Bun(C)_{r,d}$, use Faltings approach

The valuative criterion

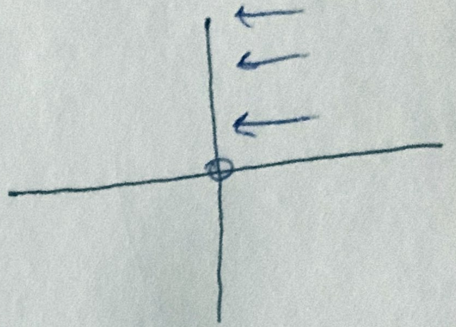
Θ-reductivity Set $\Theta = [\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{gms} \operatorname{Spec} k$

For a DVR R, set

$$\begin{array}{ccc} \Theta_R = \Theta \times \operatorname{Spec} R & \longrightarrow & [\mathbb{A}^2/\mathbb{G}_m] \quad \text{weights } 1,0 \\ \downarrow{\scriptstyle gms} & & \downarrow{\scriptstyle gms} \\ \operatorname{Spec} R & \longrightarrow & \mathbb{A}^1 \end{array}$$

Set $K = \operatorname{Frac}(R)$
$\pi \in R$ unif.

Since $\Theta_R = \operatorname{Spec} R[x]/\mathbb{G}_m$,



$$\Theta_R \setminus 0 = \Theta_R \setminus V(x,\pi) = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \Theta_R$$

Defn An alg. stack w/ affine diag $\mathcal{X}$ is **Θ-reductive** if

$$\forall \quad \begin{array}{ccc} \Theta_R \setminus 0 & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow \exists & \\ \Theta_R & & \end{array}$$

~~What does this mean geometrically?~~ What are maps from Θ?

Ex 1 $\Theta \to [X/G] \longleftrightarrow x \in X(k)$ & $\lambda : \mathbb{G}_m \to G$ s.t. $x_0 = \lim_{t \to 0} \lambda(t) x$ exists

$1 \longmapsto x$
$0 \longmapsto x_0$

Ex 2 $\Theta \to \operatorname{Bun}(C)_{r,d} \longleftrightarrow$ vect. bdl E & R-filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n$

$1 \longmapsto E$
$0 \longmapsto \bigoplus_i E_i/E_{i-1}$

Reason:

$\mathbb{G}_m$-equiv $\mathbb{F}$ $\qquad$ $\mathbb{F}$ / flat/Θ $\qquad$ $\mathbb{F}$ is an $\mathcal{O}_C[x]$-module with $\mathbb{G}_m$-action $\rightsquigarrow F = \bigoplus F_d$ with $x : F_d \to F_{d+1}$

$$C \longleftarrow C \times \mathbb{A}^1 \longrightarrow C \times \Theta$$

(4)

Ex 3 $\Theta \to \overline{M}_g \longleftrightarrow$ $\mathbb{G}_m$-equiv family

or any moduli stack of varieties

$$\begin{array}{c} \mathbb{G}_m \curvearrowright C_0 \\ \downarrow \\ \mathbb{G}_m \curvearrowright \mathbb{A}^1 \end{array}$$

Fact: Any DM stack is Θ-reductive

$\rightsquigarrow$ "test configuration"

What is Θ-reductivity geometrically?

(1) For $[X/G]$,

Θ-reductivity $\iff$ $\forall\, \operatorname{Spec} R \xrightarrow{f} X$ & $\lambda: \mathbb{G}_m \to G$

limit exists generically $\Rightarrow$ limit of $f(\lambda)$ exists

Lemma 1 BGL_n is Θ-reductive

Pf

$$\begin{tikzcd} E \curvearrowright \Theta_R \setminus 0 \arrow[r] \arrow[d, "j"] & BGL_n \text{ vect. bdl} \\ j_* E \text{ vect. bdl} \curvearrowright \Theta_R \arrow[ur, dotted] \end{tikzcd}$$

Lemma 2 If $f: \mathcal{X} \to \mathcal{Y}$ affine, $\mathcal{Y}$ Θ-reductive $\Rightarrow$ $\mathcal{X}$ Θ-reductive

Pf

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r, "h"] \arrow[d, "j"] & \mathcal{X} \arrow[d, "f"] \\ \Theta_R \arrow[r, dashed, "g"] & \mathcal{Y} \end{tikzcd}$$

$j_* \mathcal{O}_{R \setminus 0} = \mathcal{O}_R$

$$\begin{aligned} \operatorname{Map}_{\mathcal{Y}}(\Theta_R \setminus 0, \mathcal{X}) &= \operatorname{Hom}_{\mathcal{O}_{\mathcal{Y}}\text{-alg}}(f_* \mathcal{O}_{\mathcal{X}}, (g \circ j)_* \mathcal{O}_{R \setminus 0}) \\ &= \operatorname{Hom}_{\mathcal{O}_{\mathcal{Y}}\text{-alg}}(f_* \mathcal{O}_{\mathcal{X}}, g_* \mathcal{O}_R) \\ &= \operatorname{Map}_{\mathcal{Y}}(\Theta_R, \mathcal{X}) \end{aligned}$$

Lemma 3: G reductive $\Rightarrow$ $[\operatorname{Spec} A/G]$ Θ-reductive

Pf Suffices to show BG Θ-reductive (as $[\operatorname{Spec} A/G] \to BG$ affine)

Choose $G < GL_n$. Then GL_n/G affine

$BG = [(GL_n/G)/GL_n]$ ✓

② For $\mathrm{Bun}(C)_{r,d}$,

Θ-reductivity $\iff$ Given vect. bdl E on C_R, any filtration of E_K extends to E.

Properness of Quot ✓

For $\mathrm{Bun}(C)^{ss}_{r,d}$, if factors $(E_K)_n/(E_K)_{n-1}$ ss (E ss & of same slope) $\Rightarrow$ $(E_0)_n/(E_0)_{n-1}$ ss

Reason: deg & rk are constant in flat families

③ For moduli of Fano varieties,

$$\begin{tikzcd} K_X \ \ \mathcal{X} \arrow[d, "\pi"] \\ \Theta_{R,0} \arrow[r, hook, "j"] & \Theta_R \end{tikzcd}$$

$\overline{\mathcal{X}} = \underbrace{\mathrm{Proj}_{\Theta_R} \bigoplus_{d\geq 0} j_*\pi_*(K_X^{-d})}$

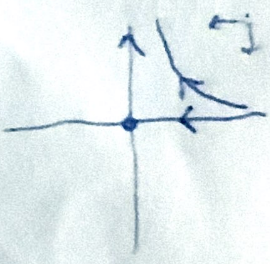
filtr gen? MMMP

S-completeness Recall R DVR, $K = \mathrm{Frac}(R)$, $\pi \in R$ unif. param

Define

weights $1,-1$

$$\begin{tikzcd} 0 = V(x,y) \in \phi_R = \left[\mathrm{Spec}(R[x,y]/(xy-\pi))/\mathbb{G}_m\right] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^2/\mathbb{G}_m]} \arrow[d] \\ \mathrm{Spec} R \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$



Observe that $\phi_R \setminus 0 = \mathrm{Spec} R \cup_{\mathrm{Spec} K} \mathrm{Spec} R$

$$\begin{tikzcd} \mathrm{Spec} R \arrow[dr, "x \neq 0"] & & \Theta \arrow[dl, "x=0"'] \\ & \phi_R & \\ \mathrm{Spec} R \arrow[ur, "y \neq 0"'] & & \Theta \arrow[ul, "y=0"] \end{tikzcd}$$

<u>Defn</u> We say an alg stack $\mathcal{X}$ w/ affine diag is <u>S-complete</u> if

$$\begin{tikzcd} \forall\ \operatorname{Spec}R \cup_{\operatorname{Spec}K} \operatorname{Spec}R = \Theta_R \setminus 0 \arrow[r] \arrow[d, hook] & \mathcal{X} \\ \Theta_R \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

<u>Fact</u> For DM stack, $\mathcal{X}$ S-complete $\iff$ $\mathcal{X}$ separated

Reason: ↳ Any map

$$\begin{tikzcd} \operatorname{Spec}R \cup_{\operatorname{Spec}K} \operatorname{Spec}R \arrow[r] \arrow[d] & \mathcal{X} \\ \operatorname{Spec}R \arrow[ur, dashed] & \end{tikzcd}$$

ordinary valuative

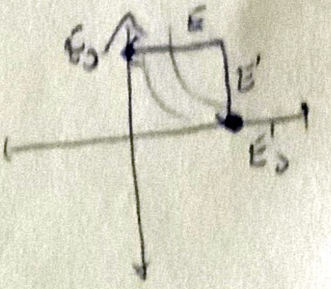
↳ Any map

$$\begin{tikzcd} \Theta_R \arrow[r] \arrow[d] & \mathcal{X} \leftarrow \text{DM} \\ \operatorname{Spec}R \arrow[ur, dashed] & \end{tikzcd}$$

<u>Example</u> For a moduli problem $\mathcal{M}$, S-completeness means

Given $E, E' \in \mathcal{M}(R)$ w/ $E_K \cong E'_K$ extends over Θ_R

$\Rightarrow \exists\ \Theta \xrightarrow{h} \mathcal{M},\ \Theta \xrightarrow{h'} \mathcal{M}$
$1 \mapsto E_0 \qquad 1 \mapsto E'_0$
s.t. $h(0) = h'(0)$



<u>Example</u> $\operatorname{Bun}(C)^{ss}_{r,d}$ is S-complete

In particular, any 2 extensions of a ss. vect bdl $E^{\times}$ on C_K are <u>S-equivalent</u>

"S" is for Seshadri

<u>Fact</u> $[\operatorname{Spec} A / G]$ is S-complete (same argument as Θ-reductive)
$\uparrow$ reductive

<u>Fact</u> BG S-complete $\iff$ G reductive.

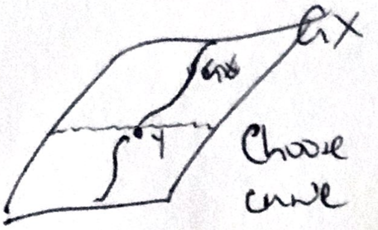
<u>Rmk</u>: Related to Hilbert-Mumford criterion

Hillbert-Mumford criterion

G reductive$/k=\bar{k} \curvearrowright X=\operatorname{Spec} A$ f.type$/k$

For $x\in X(k)$, $\exists\, \lambda: \mathbb{G}_m \to G$ s.t. $\lim_{t\to 0}\lambda(t)x$ is in the unique closed orbit in $\overline{Gx}$.

Pf



Choose curve

Let $y\in\overline{Gx}$ have closed orbit

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "g"] \arrow[d] & G \arrow[r] & \overline{Gx} \arrow[d] & h \mapsto hx \\ \operatorname{Spec} R \arrow[rr] & & X & \\ 0 \arrow[rr, mapsto] & & y & \end{tikzcd}$$

$R=k[[t]]$ DVR

Suffices to show

Iwahori decomp: $\forall\, g\in G(K)\ \exists\, h_1,h_2\in G(R)$ & $\lambda:\mathbb{G}_m\to G$ s.t. $g=h_1\lambda|_K h_2\in G(K)$.

$g\in G(K)$ determines $\operatorname{Spec} R\cup_{\operatorname{Spec} K}\operatorname{Spec} R\to BG$ by gluing trivial G-torsors w/ g.

$\exists$ exists b/c G-reductive (next time)

$$\begin{tikzcd} & \operatorname{Spec} R\cup_{\operatorname{Spec} K}\operatorname{Spec} R \arrow[d] \\ {[\mathbb{A}^2/_{1,-1}\mathbb{G}_m]} \arrow[d] & {[\operatorname{Spec}(R[x,y]/(xy-\pi))/\mathbb{G}_m]} \arrow[l] \arrow[d, "gms"] \arrow[dl, phantom, "\lrcorner"] \\ \mathbb{A}^1 & \operatorname{Spec} R \arrow[l] \end{tikzcd}$$

$t\cdot(x,y)=(tx,t^{-1}y)$

$$\begin{tikzcd} \rightsquigarrow \operatorname{Spec} R\cup_{\operatorname{Spec} K}\operatorname{Spec} R \arrow[r, "g\in G(K)"] \arrow[d] & BG & \\ \mathcal{X}=[\operatorname{Spec} R[x,y]/(xy-\pi)/\mathbb{G}_m] \arrow[ur, "\tilde{g}"] \arrow[r, shift left, "\pi"'] & B\mathbb{G}_m=\mathcal{X}_0 \arrow[l, shift left, "i"'] \arrow[u, "\lambda"'] \arrow[ul, phantom, "\lambda\circ\pi"] & \end{tikzcd}$$

gives $\lambda: \mathbb{G}_m\to G$

i.e. surjective on morphisms

$$\operatorname{Map}(\mathcal{X},BG)\overset{cc}{\simeq}\varprojlim \operatorname{Map}(\mathcal{X}_n,BG)\xrightarrow{\text{full}}\operatorname{Map}(B\mathbb{G}_m,BG)$$

def theory

$\tilde{g}\ \ \lambda\circ\pi$

Conclusion $\tilde{g} \simeq \lambda \circ \pi$

$\Longrightarrow$ g and $\lambda|_K \in G(K)$ define equivalent maps

$$\operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \longrightarrow BG$$

$\Longrightarrow$ $g = h_1 \lambda|_K h_2$ for some $h_1, h_2 \in G(R)$