Cox rings with Applications in Algebraic Geometry

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Lecture 1 Definitions, Basic Properties and Examples

Cox sheaves and Cox rings, finite gen. and UFD.

X alg. variety $/\mathbb{K}$, $\mathbb{K} = \overline{\mathbb{K}}$, char $\mathbb{K} = 0$ $(\mathbb{K} = \mathbb{C})$

<u>normal</u>: $\mathcal{O}_{x,X}$ integrally closed

prime divisor: $D \subseteq X$ irreducible subvar. codim $= 1$

Weil divisors: $\underset{\uparrow \text{prime}}{\sum a_i D_i}$, $a_i \in \mathbb{Z} \rightsquigarrow WDiv(X)$

D <u>effective</u> if $a_i \geq 0$ $(D \geq 0) \rightsquigarrow WDiv(X)^+$

$$f \in \mathbb{K}(X) \setminus \{0\} \rightsquigarrow div(f) = \sum_{D \text{ prime}} \nu_D(f) D = \underset{\uparrow \text{zeroes}}{D_+} - \underset{\uparrow \text{poles}}{D_-}$$

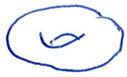
$$PDiv(X) = \{div(f) \mid f \in \mathbb{K}(X) \setminus \{0\}\} \leq WDiv(X)$$

$$Cl(X) := WDiv(X) / PDiv(X)$$

<u>Examples</u>

1) X affine: $Cl(X) = 0 \Leftrightarrow \mathbb{K}[X]$ UFD

2) $X = \mathbb{P}^n$: $Cl(\mathbb{P}^n) = \langle [H] \rangle \cong \mathbb{Z}$

3) $X =$  : $Cl(X) \cong$  $\oplus \mathbb{Z}$

elliptic curve

$$D \in WDiv(X) \rightsquigarrow H^0(X, D) := \{f \in \mathbb{K}(X)^\times : div(f) + D \geq 0\} \cup \{0\}$$

<u>Assume that</u> $Cl(X) \cong \mathbb{Z}^m$ and fix $K \leq WDiv(X)$: $K \xrightarrow{\sim} Cl(X)$

<u>Cox ring</u>: $R(X) = \bigoplus_{D \in K} H^0(X, D)$, $R(X)_0 = \mathbb{K}[X]$

$$f_1 \in H^0(X, D_1),\ f_2 \in H^0(X, D_2) \Rightarrow f_1 \cdot f_2 \in H^0(X, D_1 + D_2)$$

$R(X) \subseteq \mathbb{K}(X)[T_1^{\pm 1}, \ldots, T_m^{\pm 1}]$, chang $K \Leftrightarrow D_i \mapsto D_i + div(h_i)$

$\Leftrightarrow T_i \mapsto T_i h_i^{-1}$

$\Rightarrow R(X)$ does not depend on K

Examples 1) $R(\mathbb{A}^n) = \mathbb{K}[\mathbb{A}^n] = \mathbb{K}[x_1, \dots, x_n]$, $\deg(x_i) = 0$

2) $R(\mathbb{P}^n) = \mathbb{K}[z_0, z_1, \dots, z_n]$, $\deg(z_i) = 1$

More generally, assume that $Cl(X)$ is fin. gen, i.e. $Cl(X) \cong \mathbb{Z}^m \oplus \underset{\uparrow \text{ finite abelian}}{A}$, $\exists\, WDiv(X) \supseteq \overset{\text{f.g.}}{K} \twoheadrightarrow Cl(X)$

$\Rightarrow R(X) = \bigoplus_{D \in K} H^0(X, D) \big/ \mathcal{I}$, $H^0(X, D) \leftrightarrow H^0(X, D')$ iff $[D] = [D']$

Formally, $0 \to K^0 \to K \to Cl(X) \to 0 \Rightarrow$

$\exists\, \chi : K^0 \to \mathbb{K}(X)^\times$, $\underset{K_0}{\overset{\cap}{E}} \mapsto \chi(E)$, $\operatorname{div}(\chi(E)) = E$

and $\mathcal{I} = (\underset{H^0(X,0)}{\overset{\cap}{1}} - \underset{H^0(X,-E)}{\overset{\in}{\chi(E)}})$

To check well-definedness we need $\mathbb{K}[X]^\times = \mathbb{K}^\times$.

So again $R(X) = \bigoplus_{u \in Cl(X)} R(X)_u$ $\left\{ \begin{array}{l} \text{integral} \\ \text{normal} \end{array} \right.$

Unique factorization Let $R = \bigoplus_{u \in F} R_u$ be a graded algebra. It is *factorially graded* if every homog. nonzero nonunit $f \in R_u$ is a product of homog. primes.

Lemma 1) If $R^\times = \mathbb{K}^\times$, then R UFD $\Rightarrow$ R fact. graded

2) If $F \cong \mathbb{Z}^m$, then the converse holds.

Geometric arguments for 2): $\mathbb{Z}^m$-grading $\Leftrightarrow$ $\underset{\text{torus action}}{T^m \circlearrowright X = \operatorname{Spec} R}$

factorially graded $\Leftrightarrow$ any T^m-invar. divisor is principle. But any Weil divisor is $\sim$ to a T-inv. divisor

Thm $R(X)$ is factorially graded.

$\square$ $WDiv(X)^+$ is freely generated by prime D_i; so it is factorial.

But any $D \in WDiv(X)^+ \leftrightarrow$ line in $H^0(X, D')$ with $[D] = [D'] \in Cl(X)$

So for the monoid $HR(X)$ of homog. elements in $R(X)$ we have $HR(X)/\mathbb{K}^\times \cong WDiv(X)^+$: graded factoriality $\blacksquare$

Corollary $Cl(X) \cong \mathbb{Z}^m \Rightarrow R(X)$ is UFD

If $Cl(X)$ has torsion, then $R(X)$ sometimes is UFD, sometimes not.

$R(X)$ is not always finitely generated:

Reason 1: $\exists$ factorial quasiaff. X: $\mathbb{K}[X] = R(X)$ not f.g.

Example $X = SL_{18}/H$ counterexample to Hilbert's 14th Problem
(H unipotent)

Reason 2: $R(X) = \bigoplus_{u \in Cl(X)} R(X)_u = \bigoplus_{u \in Eff(X)} R(X)_u$ and cone$(Eff(X))$ may be not polyhedral

Example $X = Bl_{p_1 \dots p_s}(\mathbb{P}^2)$, $s \geq 9$, $p_1, \dots, p_s$ generic points
$\Rightarrow$ infinitely many (-1)-curves, whose classes generate infinitely many rays of cone$(Eff(X))$.

Positive results:

X is toric if $\exists\, T \curvearrowright X$ with an open orbit $\mathcal{O} \subseteq X$

Then $X \setminus \mathcal{O} = D_1 \cup \ldots \cup D_m$

$\uparrow$ T-invariant prime divisors

Thm 1) (Cox '95) X toric $\Rightarrow$

$\Rightarrow R(X) = \mathbb{K}[x_1, \ldots, x_m]$, $\deg(x_i) = [D_i] \in Cl(X)$

polynomial algebra

2) X complete and $R(X) = \mathbb{K}[x_1, \ldots, x_m]$

$\Rightarrow$ X toric

Remark Assertion 2) is open when X is affine, it is connected with the linearization problem for tori.

Lecture 2 **Remark** If $\mathcal{U} \underset{\text{open}}{\subseteq} X$ and $\mathcal{U} \cong \mathbb{A}^n$, then $X \setminus \mathcal{U} = D_1 \cup \dots \cup D_s$ and the classes $[D_1], \dots, [D_s]$ generate $Cl(X)$ freely.

Example Take $\mathbb{P}^1$, $A = (a_0, \dots, a_r)$ pairwise distinct points on $\mathbb{P}^1$,
$n = (n_0, \dots, n_r)$, $n_i \in \mathbb{Z}_{\geq 2}$

$\rightsquigarrow$ prevariety $\mathbb{P}^1(A,n)$ smooth, ~~irred.~~, rational, dim $= 1$

$X_{ij} = \mathbb{P}^1 \setminus \bigcup_{k \neq i} a_k$, $0 \leq i \leq r$, $1 \leq j \leq n_i$

$\underset{X}{\overset{\parallel}{\mathbb{P}^1(A,n)}}$ gluing of X_{ij} along $\mathbb{P}^1 \setminus A$

$R(X) = ?$

Step 1 $Cl(X) = \bigoplus_{j=1}^{n_0} \mathbb{Z}[a_{0j}] \oplus \bigoplus_{i=1}^{r} \left(\bigoplus_{j=1}^{n_i - 1} \mathbb{Z}[a_{ij}] \right)$

Step 2 $R(X)$ is generated by $T_{ij} \leftrightarrow 1 \in H^0(X, a_{ij})$

Step 3 If $r \leq 1$ then X toric $\Rightarrow R(X) = \mathbb{K}[T_{ij}]$

Step 4 If $r \geq 2$ then $T_i := T_{i1} \cdots T_{in_i}$, $a_i = [b_i : c_i] \in \mathbb{P}^1$

$$R(X) = \mathbb{K}[T_{ij}] / (g_s), \quad g_s = (b_{s+1} c_{s+2} - b_{s+2} c_{k+1}) T_s + (b_{s+2} c_s - b_s c_{s+2}) T_{s+1} + (\dots) T_{s+2}, \quad 0 \leq s \leq r-2$$

Cox rings and GIT: X normal, $Cl(X)$ f.g., $\mathbb{K}[X]^\times = \mathbb{K}^\times$

Cox sheaf $\mathcal{R}$: $\mathcal{U} \underset{\text{open}}{\subseteq} X \rightsquigarrow \mathcal{R}(\mathcal{U}) = \bigoplus_{D \in K} H^0(\mathcal{U}, D) \big/ \mathcal{I}(\mathcal{U})$

Claim $R(X)$ is the ring of global sections of $\mathcal{R}$.

Geometric objects : $\hat{X} = \mathrm{Spec}_X \mathcal{R}$

$\uparrow$ characteristic space

<u>Claim</u> If X is $\mathbb{Q}$-factorial, then $\mathcal{R}$ is locally of finite type and $\hat{X}$ is quasiaffine, $\mathbb{K}[\hat{X}] = R(X)$

<u>Def</u> If $R(X)$ is finitely gen., then X is a Mori Dream Space (MDS)

In this case we have $\overline{X} := \mathrm{Spec}\, R(X)$

$\uparrow$ total coord. space: affine (factorial) variety

Finally,

$H := \mathrm{Spec}\, \mathbb{K}[Cl(X)]$ diagonalizable group

$\uparrow$ characteristic quasitorus $T \times A$

<u>Thm</u> We have

$$\begin{array}{ccl} \hat{X} & \xhookrightarrow[\text{open}]{} & \overline{X} \; : \; \mathrm{codim}_{\overline{X}}(\overline{X} \setminus \hat{X}) \geq 2 \\ \downarrow /\!\!/ H & & \text{good quotient} \\ X & & \end{array}$$

<u>Properties:</u>

1) $H \curvearrowright \hat{X}$ freely $\Leftrightarrow$ X locally factorial (e.g. smooth)

$\hat{X} \xrightarrow{/H} X$ <u>universal torsor</u>

2) $\hat{X} \xrightarrow{/H} X$ is geometric

$\Leftrightarrow$ all stabilizers are finite $\Leftrightarrow$ X is $\mathbb{Q}$-factorial

3) $\hat{X} = \overline{X} \Leftrightarrow X$ affine

<u>Example</u> $\mathbb{A}^{n+1} \setminus \{0\} \hookrightarrow \mathbb{A}^{n+1}$

$$\begin{array}{c} \mathbb{A}^{n+1} \setminus \{0\} \\ \downarrow /\mathbb{K}^{\times} \\ \mathbb{P}^n \end{array}$$

<u>Toric case</u> :

$$\begin{array}{ccc} \mathbb{A}^m \setminus Z & \subseteq & \mathbb{A}^m, \\ \downarrow /\!\!/ H & & \\ X & & \end{array}$$

Z union of some coordinate planes in $\mathbb{A}^m$ of codim ≥ 2

Def X toric is <u>non-degenerate</u> if $\mathbb{K}[X]^\times = \mathbb{K}^\times$ (3)

$\iff X \not\cong \underset{\text{toric}}{X'} \times T_1$

<u>Observation</u> X toric aff. non-degenarate

$\iff X \cong \mathbb{A}^m /\!\!/ H$, $H \curvearrowright \mathbb{A}^m$ linearly.

<u>Linearization Problem</u> for tori: $T \curvearrowright \mathbb{A}^m$

$\overset{?}{\Rightarrow}$ this action is conjugate in $\mathrm{Aut}(\mathbb{A}^m)$ to a linear action.

Yes for $m \leq 3$, for $m=4$ and $\begin{matrix}\mathbb{K}^\times \\ (\mathbb{K}^\times)^2\end{matrix} \curvearrowright \mathbb{A}^4$ open

<u>Algebraic reformulation</u>: $\mathbb{K}[x_1, \dots, x_m] = A = \bigoplus_{u \in \mathbb{Z}^k} A_u$

Can we find homog. alg. indep. generators?

<u>Connection</u>: X aff., $\overline{X} = \mathbb{A}^m \Rightarrow X = \mathbb{A}^m /\!\!/ H$

If $H \curvearrowright \mathbb{A}^m$ is linear then X is toric

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<u>Iteration of Cox rings</u>: $\overline{X} \supseteq \hat{X} \xrightarrow{/\!\!/ H} X$. If $Cl(X)$ has torsion, then $\overline{X}$ may be not factorial $\Rightarrow$

$\Rightarrow$ take the Cox ring of $\overline{X}$, $X_1 = \mathrm{Spec}\, R(\overline{X})$

and so on: $\dots X_s \xrightarrow{/\!\!/ H_s} \dots \to X_1 \xrightarrow{/\!\!/ H_1} \overline{X} \supseteq \hat{X} \xrightarrow{/\!\!/ H} X$

It terminates if whether $Cl(X_s)$ is not fin. gen., or $Cl(X_s) = 0$. Always terminates?

Given a quotient presentation $\underset{\text{quasiaf.}}{\hat{X}} \xrightarrow{/\!/H} X$,

when is it canonical (Cox)? Lecture 3

Thm This is the case iff

$\mathbb{K}[\hat{X}]^\times = \mathbb{K}^\times$, $\hat{X}$ is H-factorial (all H-inv. divisors are principle)

and $H \curvearrowright \hat{X}$ strongly stable:

$\exists$ open H-inv. $W \subseteq \hat{X}$:
1) $\operatorname{codim}_{\hat{X}}(\hat{X} \setminus W) \geq 2$;
2) $H \curvearrowright W$ freely;
3) $\forall x \in W$ Hx is closed in $\hat{X}$.

($*$ W = preimage of X^{reg}_*)

Example Take $\underset{G}{SL_n} \supseteq B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \supseteq U = \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\}$

$X = G/B$ flag variety, smooth, projective. $R(X)$ - ?

$\mathbb{K}[SL_n]$ UFD $\Rightarrow$ $\mathbb{K}[SL_n]^U$ UFD $\Rightarrow$

$\Rightarrow Cl(SL_n/U) = 0$ and $\mathbb{K}[SL_n]^\times = \mathbb{K}^\times$

↑ quasiaffine

$T = \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \subseteq SL_n$, $T \curvearrowright SL_n/U$, $t \circ gU = gt^{-1}U$

and $(SL_n/U)\overset{\text{free}}{/}T \cong G/B \Rightarrow \widehat{G/B} = G/U$ and $R(G/B) = \mathbb{K}[G]^U$.

More generally, G semisimple simply connected, $F \subseteq G$ closed subgroup, $X = G/F$ quasiprojective

Then 1) $Cl(X) \cong \mathfrak{X}(F)$. Let $F_1 := \bigcap_{\chi \in \mathfrak{X}(F)} \operatorname{Ker} \chi$

2) $\hat{X} = G/F_1 \xrightarrow{/H} G/F$, where $H := F/F_1$.

Let us recall from Lecture 1

Reason 2 for non-fin. generation of $R(X)$:

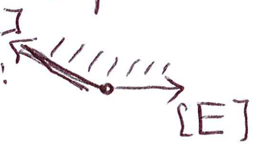
if ~~cone~~ $(\mathrm{Eff}(X))$ is not polyhedral, then $R(X)$ is not f.g.

<u>Question</u> $\mathrm{Eff}(X)$ polyhedral $\overset{?}{\Longrightarrow}$ $R(X)$ fin. gen.

Sometimes yes: say, for K3-surface

But in general no even for surfaces

<u>Example</u> Let S be a smooth general quartic surface in $\mathbb{P}^3$ and $p \in S$ be a general point. Let $X = \mathrm{Bl}_p S$.

$\mathrm{Eff}(X)$: 

C is nef, but not semiample

$\Rightarrow X$ is not Mori Dream Space

<u>Types of divisors</u> : (1) effective

<u>Picture 1</u>

Base locus: $Bs(D) = \bigcap\limits_{\substack{D' \sim D \\ \text{eff}}} \mathrm{Supp}(D')$

stable base locus: $B(D) = \bigcap\limits_{n \geq 0} Bs(nD)$

(2) D <u>movable</u> if $\mathrm{codim}_X B(D) \geq 2$

(3) D <u>numerically effective</u> (nef) if $D \cdot C \geq 0$ $\forall C$ irr. curve

(4) D <u>semiample</u> if $B(D) = \emptyset$

(5) D <u>ample</u> if $X = \bigcup\limits_i \mathcal{U}_i$ and $X \setminus \mathcal{U}_i = \mathrm{Supp}(D_i)$, $\mathcal{U}_i$ open affine, $D_i \sim nD$ for some n.

<u>Orbit cones and GIT-chambers</u>

<u>Picture 2</u>

$\underset{\text{torus}}{H} \curvearrowright \underset{\text{aff}}{Z = \overline{X}}$, $\mathbb{K}[Z] = \mathbb{K}[f_1, \dots, f_k]$, $h \cdot f_i = \chi_i(h) f_i$

$\chi_i \in \mathfrak{X}(H) \cong \mathbb{Z}^m$

Weight cone $\omega_Z = \mathrm{cone}(\chi_1, \dots, \chi_k) \subseteq \mathbb{Q}^m$

Let us assume that ω_Z is pointed

Take $x \in Z$ and define orbit cone $\omega_x = \mathrm{cone}\left(\chi \,\middle|\, \exists f_\chi : f_\chi(x) \neq 0\right)$

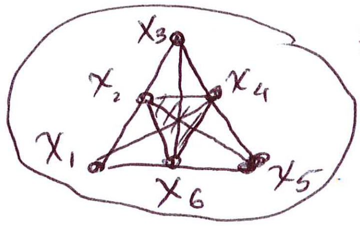
It is generated by a subset of $\{\chi_1, \dots, \chi_k\}$

Take $\chi \in \omega_Z$ and define its GIT-chamber as

$$\lambda(\chi) = \bigcap_{\chi \in \omega_x} \omega_x$$

Claim GIT-chambers form a fan $\{\lambda(\chi), \chi \in \omega_Z\}$ with ω_Z as the support. It is called GIT-fan.

Example $\omega_Z \subseteq \mathbb{Q}^{m=3}$, $\chi_1, \dots, \chi_6 \in V(z_3 = 1)$



$z_3 = 1$

$$\mathrm{Mov}_Z := \bigcap_{i=1}^{k} \mathrm{cone}(\chi_1, \dots, \hat{\chi_i}, \dots, \chi_k)$$

moving cone

$\forall \chi \in \omega_Z$ we define $Z^{ss}(\chi) = \left\{ x \in Z \,\middle|\, \begin{array}{l} \exists f_{k\chi}, k \in \mathbb{Z}_{>0} \\ f_{k\chi}(x) \neq 0 \end{array} \right\}$

Then we have $X(\chi) := Z^{ss}(\chi) /\!\!/ H$

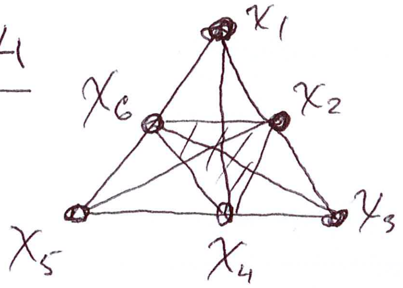
Claim If $\lambda(\chi) = \lambda(\chi')$ then $Z^{ss}(\chi) = Z^{ss}(\chi')$

and so $X(\chi) = X(\chi')$

Lecture 4

Moving cone

$H \curvearrowright Z$



$\mathbb{K}[Z] =$

$= \mathbb{K}[f_1, \dots, f_k]$
$\chi_1, \dots, \chi_k$

$$\mathrm{Mov}_Z := \bigcap_{i=1}^{k} \mathrm{cone}(\chi_1, \dots, \hat{\chi}_i, \dots, \chi_k)$$

$$\forall \chi \in \omega_Z \Rightarrow Z^{ss}(\chi) = \left\{ x \in Z \;\middle|\; \begin{array}{l} \exists f_{k\chi},\ k>0 \\ f_{k\chi}(x) \neq 0 \end{array} \right\}$$

$$\Rightarrow \exists\ X(\chi) := Z^{ss}(\chi) /\!\!/ H \text{ projective over } Z /\!\!/ H$$

Claim $\lambda(\chi) \subseteq \lambda(\chi') \Longleftrightarrow$

$$\begin{tikzcd} Z^{ss}(\chi) \arrow[d, "/\!\!/ H"] & \supseteq & Z^{ss}(\chi') \arrow[d, "/\!\!/ H"] \\ X(\chi) & & X(\chi') \arrow[ll] \end{tikzcd}$$

Take X projective, $\mathbb{Q}$-fact, MDS, $Z = \overline{X} = \mathrm{Spec}\, R(X)$

$\Rightarrow X = X(\chi)$, $\lambda(\chi) = \mathrm{SAmple}(X) \subseteq \mathrm{Mov}_Z^{\circ}$ fulldim.

Take $D \in \mathrm{WDiv}(X)^{+}$ and let $X \dashrightarrow \mathbb{P}^n$ be defined by linear system $|kD|$, $k \gg 0$.

Let $X(D) :=$ closure of the image of ~~X~~ in $\mathbb{P}^n$

Then $X(D) = X(\mu)$, $\mu = [D]$ and $X \dashrightarrow X(D)$ is

- birational $\Longleftrightarrow [D] \in \mathrm{Eff}(X)^{\circ}$
- small modification $\Longleftrightarrow [D] \in \mathrm{Mov}(X)^{\circ}$
  (i.e. iso in codim 1)
- morphism $\Longleftrightarrow [D] \in \mathrm{SAmple}(X) = \lambda(\chi)$
- isomorphism $\Longleftrightarrow [D] \in \mathrm{Ample}(X) = \lambda(\chi)^{\circ}$

Non-projective X $\longleftrightarrow$ "bunches" of orbit cones, Gale dual of fans
(with A_2-property: $\forall x,y\in X\ \exists\, \underset{\text{open aff}}{U\subseteq X}$ s.t. $x,y\in U$)

Let G be reductive, $B\subset G$ Borel, $G \circlearrowright X$

Complexity $c_G(X) := \min\limits_{x\in X} \operatorname{codim}_X Bx = \operatorname{tr.deg} \mathbb{K}(X)^B$

Example $G=B=T$

Thm (Knop'93) X normal unirat., $G\circlearrowright X$, $c_G(X)\le 1$
$\Rightarrow \mathbb{K}[X]$ f.g.

Corollary $\Rightarrow Cl(X)$ f.g. and $R(X)$ f.g.

Lifting group action: G connect. linear alg. group

$$\begin{array}{ccccc} \hat{G} & \circlearrowright & \hat{X} & \subseteq & \overline{X} \\ {\scriptstyle\text{finite}}\downarrow & & \downarrow & & \\ G & \circlearrowright & X & & \end{array}$$

Example

$$\begin{array}{ccccc} GL_n & \circlearrowright & \mathbb{A}^n\setminus\{0\} & \subseteq & \mathbb{A}^n \\ \downarrow & & \downarrow{\scriptstyle /\mathbb{K}^\times} & & \\ PGL_n & \circlearrowright & \mathbb{P}^{n-1} & & \end{array}$$

Computing automorphism groups

Thm (Ramanujam'64) X complete $\Rightarrow \operatorname{Aut}(X)^0$ alg. group

X is MDS: $\varphi\in \operatorname{Aut}(\overline{X}, H) \Leftrightarrow \varphi\in\operatorname{Aut}(\overline{X})$ and $\varphi(hx)=\tilde{\varphi}(h)\varphi(x)$ for some $\tilde{\varphi}\in\operatorname{Aut}(H)$

$\operatorname{Bir}_2(X)$: birational automorp. + iso of two big open subsets

Claim

$$\begin{array}{ccccccccc} 1 & \to & H & \to & \mathrm{Aut}(\overline{X}, H) & \to & \mathrm{Bir}_2(X) & \to & 1 \\ & & \| & & \cup & & \cup \text{ of finite index} & & \\ 1 & \to & H & \to & \mathrm{Aut}(\hat{X}, H) & \to & \mathrm{Aut}(X) & \to & 1 \end{array}$$

$\Rightarrow \mathrm{Aut}(X)^\circ$ depends only on $R(X)$.

Thm X complete MDS $\Rightarrow \mathrm{Aut}(X)$ linear alg.

$\square$ $\exists$ f. dim $\mathcal{U} \subseteq R(X)$: (1) $\mathcal{U}$ generates $R(X)$
(2) $\mathcal{U}$ is $\mathrm{Aut}(X)$-invar.

$\Rightarrow \mathrm{Aut}(X) \subseteq G(\mathcal{U})$ closed $\blacksquare$

Cox-Nagata rings

Take $a_1, \dots, a_r \in \mathbb{A}^{n+1}$ linearly indep.

$p_i = [a_i] \in \mathbb{P}^n$, $X = \mathrm{Bl}_{p_1 \dots p_r}(\mathbb{P}^n)$

Take $A = (a_1 \dots a_r)$ $(n+1) \times r$-matrix

$\mathrm{Ker}\, A = \mathcal{U} \subseteq \mathbb{K}^r$ commut. unipot. group

$\mathcal{U} \curvearrowright \mathbb{K}^r \times \mathbb{K}^r$, $u \cdot (z, w) = (z, w_1 + u z_1, \dots, w_r + u z_r)$

Thm $R(X) \cong \mathbb{K}[z_1, \dots, z_r, w_1, \dots, w_r]^{\mathcal{U}}$.

Research topics: ⓪ Compute $R(X)$

① $R(X)$ is given by specific relations $\Leftrightarrow$ X - ?

② $\mathbb{G}_a$-actions on MDS X $\Leftrightarrow$ homog. of degree zero LND's on $R(X)$

③ $\exists\, \mathbb{G}_a^n \curvearrowright X$ with an open orbit $\Leftrightarrow$ $R(X)$ - ?

④ Compute $\mathrm{Aut}(X)$:
1) X complete toric (Demazure, Cox, ...)
2) X complete ration. $c_T(X)=1$ [AHHL'14]
3) ....