Moduli space of p-Lie algebras

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Motivations

Proposition. *Let $S \longrightarrow \mathrm{Spec}(\mathbb{F}_p)$ be a base scheme. We have an equivalence of categories between :*

$\{$*Flat group schemes over S, loc. of finite presentation, of height 1*$\}$

and

$\{$*O_S-loc. free p-Lie algebras of finite rank*$\}$

Given by $G \mapsto \mathrm{Lie}(G)$.

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I. Definition and first properties

Let R be a ring of characteristic $p > 0$ and l be a loc. free Lie algebra over R of finite rank. We write :

$$\begin{aligned} \mathrm{ad} : l &\longrightarrow \mathrm{End}(l) \\ x &\mapsto (y \mapsto [x,y]). \end{aligned}$$

and $Z(l) := \ker(\mathrm{ad})$.

Definition

We say that a mapping $(\cdot)^{[p]} : l \longrightarrow l$ is a *p-mapping* if:

1) for all $x \in l$, $\mathrm{ad}_{x^{[p]}} = (\mathrm{ad}_x)^p$

2) for all $\lambda \in R$ and $x \in l$, $(\lambda x)^{[p]} = \lambda^p x^{[p]}$

3) for all $x, y \in l$, $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$

Definition

If a Lie algebra can be equipped with a p-mapping, we say that it is *restrictable*.

Examples

1. Let A be an associative R-algebra, equipped with the bracket $[x,y] \mapsto xy - yx$. Then, $x \mapsto x^p$ is a p-mapping.
2. Let l be an abelian Lie algebra. Then, $\gamma \equiv 0$ is a p-mapping on l.
3. If we take $G = \mathrm{GL}_n$, the corresponding p-mapping on the matrix space $\mathrm{Lie}(\mathrm{GL}_n) = \mathrm{M}_n(R)$ is given by $x \mapsto x^p$, and for $G = \mathbb{G}_a$, it is given by $x \mapsto 0$ on $\mathrm{Lie}(G) = R$.

Theorem (Jacobson)

Let l be a Lie algebra, free over R with basis $\{x_i\}_{i\in I}$. Let us assume that for all $i \in I$, there exists $y_i \in l$ such that $\mathrm{ad}_{x_i}^p = \mathrm{ad}_{y_i}$. Then, there exists a p-mapping $(\cdot)^{[p]} : l \to l$ such that $(x_i)^{[p]} = y_i$.

Theorem

Let l be a Lie algebra over R.

1. *Let γ_1 and γ_2 be two p-mappings on l Then $\gamma_2 - \gamma_1 : l \to Z(l)$ is Frobenius-semi-linear.*
2. *Conversely, let $\phi : l \to Z(l)$ be a Frobenius-semi-linear map, and γ_1 a p-mapping on l. Then, $\gamma_1 + \phi : l \to l$ is also a p-mapping.*

In other words, this means that $\mathrm{E} := \mathrm{Hom}_{\mathrm{Frob}}(l, Z(l))$ acts on $X := \{p\text{-mappings on } l\}$, and the action is free and transitive.

Structure of the moduli space of p-mappings

Notations and conventions:

- From now on, let S be a base scheme of characteristic $p > 0$, and let $L \longrightarrow S$ be a Lie algebras vector bundle.
- We write $[\cdot,\cdot] : L \otimes L \to L$ and $\mathrm{ad} : L \to \mathrm{End}(L)$.
- We write $Z(L) = \ker(\mathrm{ad})$.
- We write $\mathrm{Frob} : S \longrightarrow S$ for the Frobenius morphism of S.
- We write

$$\mathrm{E} := \mathrm{Hom}_{\mathrm{Frob}}(L, Z(L)) = \mathrm{Hom}(\mathrm{Frob}_S^* L, Z(L)).$$

If $Z(L)$ is a vector bundle, then so is E.

Theorem

Let us define a set-valued functor as follows:

$$X : \{S\text{-schemes}\} \longrightarrow \mathrm{Set}$$
$$T \longmapsto \{p\text{-mappings on } L \times_S T\}.$$

Then, X is representable by an affine scheme, and is a formally homogeneous space under $\mathrm{E} = \mathrm{Hom}_{\mathrm{Frob}}(L, Z(L))$.

i.e. the following map

$$\mathrm{E} \times_S X \to X \times_S X$$
$$(\phi, \gamma) \mapsto (\phi + \gamma, \gamma)$$

is an isomorphism.

Theorem

Let us suppose that the center $Z(L) \to S$ is flat. Let us define :

$$S^{\mathrm{res}} : \{S\text{-schemes}\} \longrightarrow \mathrm{Set}$$

$$T \longmapsto \begin{cases} \{\emptyset\} \text{ if } L_T \text{ is Zar-loc. restrictable over } T \\ \emptyset \text{ otherwise.} \end{cases}$$

Then: -S^{res} is representable by a closed subscheme of S.
-The structure morphism $X \to S$ factors

$$\begin{array}{ccc} X & \longrightarrow & S \\ \downarrow & \nearrow & \\ S^{\mathrm{res}} & & \end{array}$$

and $X \to S^{\mathrm{res}}$ is an affine space under $E \times S^{\mathrm{res}}$.

Theorem

Let $L \to S$ be a Lie algebra vector bundle, such that L' is a locally free subbundle of rank 1.
We define a map of vector bundles as follows:

$$\alpha : L \to \mathrm{End}(L') \simeq \mathbb{G}_a$$
$$x \mapsto (\mathrm{ad}(x)_{|L'}) \mapsto \alpha(x).$$

Then, the map

$$L \to L$$
$$x \mapsto \alpha(x)^{p-1} x$$

is a p-mapping on L.

Moduli space L_3 of Lie algebras of rank 3

Definition

We define the following moduli space:

$$\mathrm{L}_3 : \mathrm{Sch} \longrightarrow \underline{\mathrm{Set}}$$
$$T \mapsto \left\{ [\cdot,\cdot] : \mathcal{O}_T^3 \otimes \mathcal{O}_T^3 \longrightarrow \mathcal{O}_T^3 \text{ ; where } [\cdot,\cdot] \text{ is a Lie bracket} \right\}$$

Theorem

1) *The functor* L_3 *is representable by an affine flat* $\mathbb{Z}$*-scheme of finite type.*
2) *The scheme* L_3 *has two irreducible components* $\mathrm{L}_3^{(1)}$ *and* $\mathrm{L}_3^{(2)}$ *which are both flat with Cohen-Macaulay integral geometric fibers of dimension* 6*.*

Proof: Let $T \longrightarrow S$ be a scheme. Let us write $\{x, y, z\}$ for a basis of $\mathcal{O}_T^3$. In order to know a bracket on $\mathcal{O}_T^3$, we only need to know :

$$[x,y] = ax + by + cz$$
$$[x,z] = dx + ey + fz$$
$$[y,z] = gx + hy + iz$$

with $(a, \cdots, i) \in \mathcal{O}_T(T)^9$.
The Jacobi conditions are equivalent to:

$$\begin{cases} Q_1 = ah + di - fg - bg = 0 \\ Q_2 = ie + bd - fh - ae = 0 \\ Q_3 = hc + dc - af - bi = 0 \end{cases}$$

Then,

Proposition. L_3 *is representable by a closed subscheme of* $\mathbb{A}^9_{\mathbb{Z}}$*, given by*

$$\mathrm{Spec}\left(\mathbb{Z}[a,b,c,d,e,f,g,h,i]/(Q_1,Q_2,Q_3)\right).$$

Let us remark that the Jacobi condition can be written as

$$\left\{\begin{array}{l} Q_1 = 0 \\ Q_2 = 0 \\ Q_3 = 0 \end{array}\right. \Leftrightarrow \begin{pmatrix} h & -g & i \\ -e & -h & b \\ -f & -i & c \end{pmatrix} \begin{pmatrix} a-i \\ b+f \\ d+h \end{pmatrix} = 0.$$

Then let us denote

$$M := \begin{pmatrix} h & -g & i \\ -e & -h & b \\ -f & -i & c \end{pmatrix}$$

and $L_1 := a - i, L_2 := b + f$ and $L_3 := d + h$.
For the end, let us denote

$$I = (Q_1, Q_2, Q_3),\, L = (L_1, L_2, L_3),\, J = (\det(M), Q_1, Q_2, Q_3).$$

Theorem

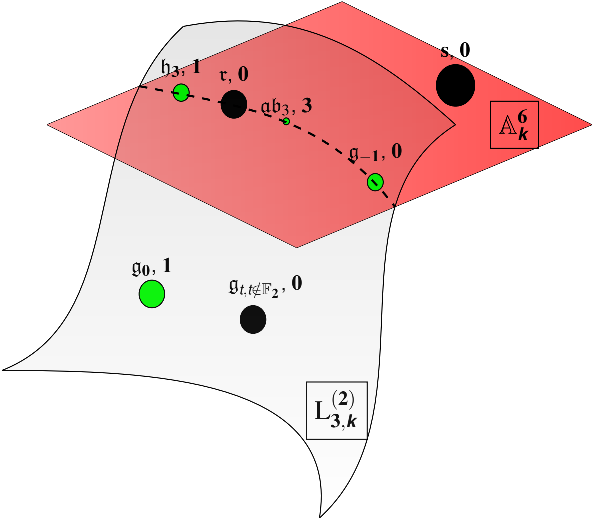
The affine scheme L_3 *can be decomposed in two irreducible components: the first one is*

$$\mathrm{L}_3^{(1)} := \mathrm{Spec}\left(\mathbb{Z}[a,\ldots,i]/L\right) \simeq \mathbb{A}^6$$

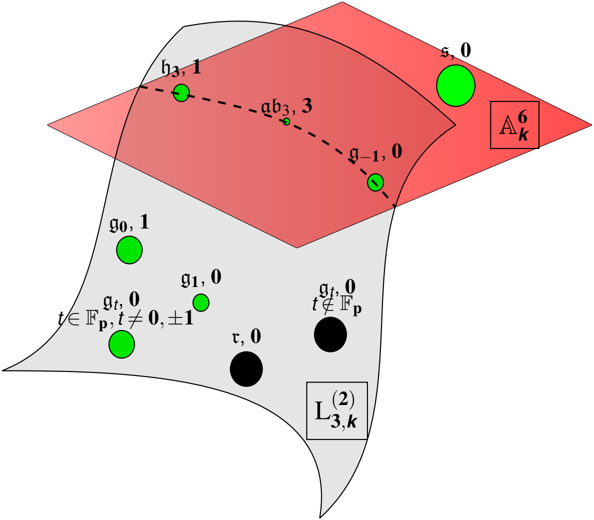
and the second one is

$$\mathrm{L}_3^{(2)} := \mathrm{Spec}\left(\mathbb{Z}[a,\ldots,i]/J\right).$$

Let $k = \overline{k}$, of characteristic 2. Here is a representation of $\mathrm{L}_{3,k}$.



Let $k = \overline{k}$, of characteristic $p \neq 2$. Here is a representation of $\mathrm{L}_{3,k}$.



Some linkage theory

Definition

Let J and L be two ideals in a ring R. We say that J and L are *linked* in R by an ideal I if $L = [I : J]$ and $J = [I : L]$.

where

$$[I : J] = \{a \in R, aJ \subset I\}.$$

Proposition.*[Peskine, Szpiro] Let R be a Gorestein local ring, and let I, L and J be ideals such that $J = [I : L]$, and $\dim(R/L) = \dim(R/I)$. Then, the following are equivalent:*

1. *R/L is Cohen-Macaulay.*
2. *R/J is Cohen-Macaulay and R/L does not have embedded components.*

In this case, $L = [I : J]$ and $\dim(R/L) = \dim(R/J)$.

Theorem

The scheme L_3 has two irreducible components $\mathrm{L}_3^{(1)}$ and $\mathrm{L}_3^{(2)}$ which are both flat with Cohen-Macaulay integral geometric fibers of dimension 6.

Back to the equivalence between groups and Lie algebras

Proposition. *Let $G \to S$ be a finite flat locally free group scheme of height 1. Let $Z(G)$ denote its center. Then*

$$Z(\mathrm{Lie}(G)) = \mathrm{Lie}(Z(G)).$$

Let k be an algebraically closed field of characteristic $p > 0$. Let us denote by $G^p_{3,r}$ the category of finite locally free k-group schemes of order p^3, of height 1, whose center is locally free of rank p^r.

Proposition. *Then $G^p_{3,r}$ splits in two irreducible components that we denote by $G^{p,(1)}_{3,r}$ and $G^{p,(2)}_{3,r}$, and we have:*

- *If $p \neq 2$, $G^p_{3,0}$ is singular, but becomes smooth after intersection with $G^{p,(2)}_{3,0}$, if $p \neq 2$, $G^p_{3,0}$ is smooth.*
- *$G^p_{3,1}$ is singular but becomes smooth when we intersect it with $G^{p,(1)}_{3,1}$.*
- *$G^p_{3,2}$ is empty and $G^p_{3,3}$ is smooth.*

References

My pre-print: Moduli of p-Lie algebra,
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Thank you for your attention !