Arithmetic Moduli of Weierstrass Fibrations

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Invariants in Algebraic Geometry

Counting Number Fields

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One could consider a one-dimensional higher analogue which is,

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A: This is the influential Shafarevich's conjecture for algebraic curves first called to attention by Igor R. Shafarevich in his 1962 address at the International Congress in Stockholm.

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Question

How many exactly are there?

Elliptic Surfaces

The study of **fibrations of curves** lies at the heart of the Enriques-Kodaira / Mumford-Bombieri classifications of compact complex surfaces / algebraic surfaces in positive characteristics.

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We call an algebraic surface X to be an **elliptic surface**, if it admits an elliptic fibration $f : X \to C$ which is a flat and proper morphism f from a nonsingular surface X to C where C is a nonsingular curve, such that a generic fiber is a smooth curve of genus one.

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While this is the most general setup, it is natural to work with the case when the base curve is the smooth projective line $\mathbb{P}^1$ and there exists a section $S : \mathbb{P}^1 \hookrightarrow X$ coming from the identity points of the elliptic fibres and not passing through the singular points.

Elliptic surfaces over $\mathbb{P}^1$ with a section

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1. When $n = 1$, X is a **Rational elliptic surface** with the Kodaira dimension $\kappa = -\infty$ which has 12 nodal singular fibers generically. It is acquired from a pencil of cubic curves in $\mathbb{P}^2$ by blowing up a base locus of nine points coming from the intersection of two general cubic curves.

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2. When $n = 2$, X is a $K3$ surface with an elliptic fibration (i.e., **Elliptic K3 surface**) which has the Kodaira dimension $\kappa = 0$ that has 24 nodal singular fibers generically.

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3. When $n \geq 3$, X is called a **Properly elliptic surface** with Kodaira dimension $\kappa = 1$ that has $12n$ nodal singular fibers generically.

Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

Let us recall that $\overline{\mathcal{M}}_{1,1}$ is a smooth proper Deligne-Mumford stack of stable elliptic curves with a coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$. This $\mathbb{P}^1$ parametrizes the j–invariants of elliptic curves.

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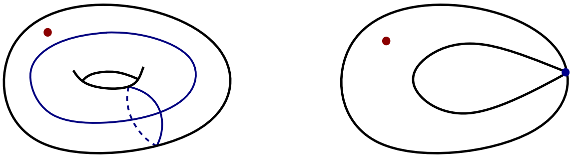

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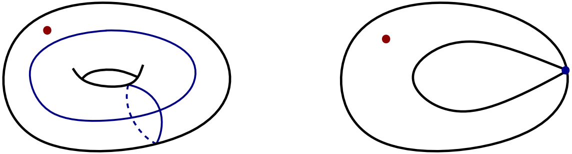

When the characteristic of the field K is not equal to 2 or 3, $(\overline{\mathcal{M}}_{1,1})_K \cong [(Spec\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] =: \mathcal{P}_K(4,6)$ through the short Weierstrass equation: $Y^2 = X^3 + a_4X + a_6$

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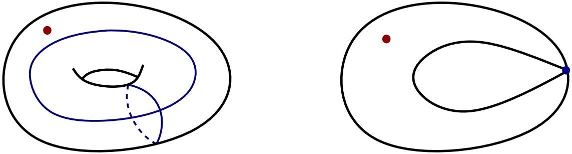

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Stabilizers are the orbifold points $[1:0]$ & $[0:1]$ with μ_4 & μ_6 respectively and the generic stacky points such as $[1:1]$ with μ_2

Moduli stack of stable elliptic surfaces

The fine moduli $\overline{\mathcal{M}}_{1,1}$ comes with universal family $p : \overline{\mathcal{C}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ of stable elliptic curves. Thus, a stable elliptic surface $g : Y \to \mathbb{P}^1$ is induced from a morphism $\varphi_f : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ and vice versa.

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Here, we *fixed the parameterization of the domain* $\mathbb{P}^1$ which is good for 'Global Fields Analogy' (since $\mathbb{Q}$ has the *unique* ring of integers called $\mathbb{Z}$) but not natural from Geometric perspective.

Group actions on stacks for stack quotients of stacks

It is natural to consider the action of $\mathrm{Aut}(\mathbb{P}^1) = \mathsf{PGL}_2$ on $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ by composing the stable elliptic surface $g : Y \to \mathbb{P}^1$ with an automorphism of $\mathbb{P}^1$.

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It is easy to see that this action is induced by an action on the ambient weighted projective stack $\mathcal{P}(V)$.

$$(A, B) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4n)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6n)) =: V \qquad (2)$$

define the so-called *Weierstrass data* of the fibration. Indeed, the action of an element of PGL_2 on the homogeneous coordinates X, Y of $\mathbb{P}^1$ translates to an action on the global sections A, B of $\mathcal{O}_{\mathbb{P}^1}(4n), \mathcal{O}_{\mathbb{P}^1}(6n)$ which are the homogeneous coordinates of $\mathcal{P}(V) = \mathcal{P}(\underbrace{4, \dots, 4}_{4n+1 \text{ times}}, \underbrace{6, \dots, 6}_{6n+1 \text{ times}}) \in \mathbb{Z}^{10n+2}$.

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Note that since both $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ and $\mathcal{P}(V)$ are themselves stacks, the formal definition of these actions requires one to use the notion of group actions on stacks presented in [Romagny].

Group actions on stacks for stack quotients of stacks

We have the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) & \hookrightarrow & \mathrm{Rat}_n^{\gamma}(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) & \hookrightarrow & \mathcal{P}(V) \\
\downarrow{\scriptstyle /\,\mathsf{PGL}_2} & & \downarrow{\scriptstyle /\,\mathsf{PGL}_2} & & \downarrow{\scriptstyle /\,\mathsf{PGL}_2} \\
\mathcal{W}_{\mathrm{sf},n} & \hookrightarrow & \mathcal{W}_{\mathrm{min},n} & \hookrightarrow & [\mathcal{P}(V)/\,\mathsf{PGL}_2]
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where the horizontal arrows are open embeddings of moduli stacks.

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Theorem (Johannes Schmitt, J.-Y. Park)

Fix a degree $n \in \mathbb{Z}_{\geq 1}$ and a base field K with $\mathrm{char}(K) \neq 2, 3$. Inside the quotient stack $[\mathcal{P}(V)/\,\mathsf{PGL}_2]$, the open substacks $\mathcal{W}_{\mathrm{min},n}$ (for $n \geq 2$) of minimal Weierstrass fibrations and $\mathcal{W}_{\mathrm{sf},n}$ (for $n \geq 1$) of stable Weierstrass fibrations are smooth, irreducible and separated Deligne–Mumford stacks of finite type with affine diagonal for $\mathrm{char}(K) \nmid n$, which are tame for $\mathrm{char}(K) > 12n$.

Group actions on stacks for stack quotients of stacks

The Weierstrass fibration associated to $[A : B] = [X^{4n} : Y^{6n}]$ is invariant under scaling X by an element of μ_{4n} and by scaling Y under an element of μ_{6n}. Together, these transformations generate a copy of μ_{12n} inside PGL_2 which acts as an automorphism of the fibration, and the quotient is not tame when $\mathrm{char}(K)$ divides $12n$.

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The stack $\mathcal{W}_{\min,1}$ contains points with positive dimensional stabilizers, thus it is no longer of Deligne–Mumford type. These points are precisely PGL_2-orbit of the Weierstrass data $[A : B]$

$$[A : B] = [0 : XY^5], [XY^3 : 0], [0 : X^2Y^4] \text{ and } [a_0X^2Y^2 : a_1X^3Y^3],$$

where in each case we have a nontrivial action of $\mathbb{G}_m$ on the coordinates X, Y fixing the fibrations. They are the four types of rational elliptic surfaces with two singular fibres $[\mathrm{II}, \mathrm{II}^*], [\mathrm{III}, \mathrm{III}^*], [\mathrm{IV}, \mathrm{IV}^*], [\mathrm{I}_0^*, \mathrm{I}_0^*]$ both of which are additive type in dual pair. One can see that the open substack $\mathcal{W}'_{\min,1}$ of $\mathcal{W}_{\min,1}$ obtained by removing these four points is indeed Deligne–Mumford for $\mathrm{char}(K) \nmid n$ and tame for $\mathrm{char}(K) > 12$.

Grothendieck ring $K_0(\mathrm{Stck}_K)$ of K-stacks

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Definition

Fix a field K. Then the *Grothendieck ring* $K_0(\mathrm{Stck}_K)$ *of algebraic stacks of finite type over* K *all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of algebraic stacks $\{\mathcal{X}\}$ modulo relations:

- $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
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Multiplication on $K_0(\mathrm{Stck}_K)$ is induced by $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times_K \mathcal{Y}\}$.

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The weighted point count of $\mathcal{X}$ over $\mathbb{F}_q$ is defined as a sum: $\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\mathrm{Aut}(x)|}$ where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of $\mathbb{F}_q$–isomorphism classes of $\mathbb{F}_q$–points of $\mathcal{X}$.

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When $K = \mathbb{F}_q$, the point counting measure $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$ gives a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$.

Motive/Point count of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ over finite fields

Theorem (Changho Han, Hunter Spink, Johannes Schmitt, J.)

The class $\{\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\}$ *in* $K_0(\mathrm{Stck}_K)$ *for* $\mathrm{char}(K) \neq 2, 3$ *of the moduli stack for stable elliptic fibrations over the parameterized* $\mathbb{P}^1$ *with* $12n$ *nodal singular fibers and a section is equivalent to*

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and over an unparameterized $\mathbb{P}^1$ *with an odd degree* n

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Then, by using $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ to count $\mathbb{F}_q$–points when $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, we acquire the weighted point counts of the moduli of stable elliptic surfaces over (un)parameterized $\mathbb{P}^1$.

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For strictly multiplicative reductions case, consider $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) :=$

$$|\{\text{Semistable elliptic curves over the } \mathbb{P}^1_{\mathbb{F}_q} \text{ with } 0 < ht(\Delta) \leq \mathcal{B}\}|$$

$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{12n}(\mathbb{F}_q)/\sim| = 2 \cdot \frac{(q^{11}-q^9)}{(q^{10}-1)} \cdot \left(\mathcal{B}^{\frac{5}{6}} - 1\right)$$

Thank you :)

Thank you to the **organizers & everyone** for listening!