HYPERKÄHLER MANIFOLDS OF $K3^{[n]}$–TYPE ADMITTING SYMPLECTIC BIRATIONAL MAPS

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Layout

Part I: Main definitions

- Hyperkähler manifolds and symplectic maps: HK manifolds of $K3^{[n]}$–type.
- Moduli spaces of (twisted) sheaves on K3 surfaces.

Part II: Problem and main results

- HK manifolds of $K3^{[n]}$–type with symplectic birational maps.
- Reflections along the vertical wall in the space of stability conditions.

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Definition

A hyperkähler manifold X is called **hyperkähler manifold of K3$^{[n]}$–type** if X is deformation equivalent to a Hilbert scheme of n points $S^{[n]}$ of some projective K3 surface S.

Theorem ([Bea83], [Fuj88])

Let X be a hyperkähler manifold of dimension $2n$. Then there exists an integral non-degenerate quadratic form q_X on $H^2(X, \mathbb{Z})$ of signature $(3, b_2(X) - 3)$ and a positive constant c_X such that

$$\int_X \alpha^n = c_X q_X(\alpha)^{2n}, \ \forall \alpha \in H^2(X, \mathbb{Z}).$$

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X	c_X	$H^2(X,\mathbb{Z})$
$S^{[n]}$	1	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus \langle -2(n-1)\rangle$
$K_n(A)$	n+1	$U^{\oplus 3} \oplus \langle -2(n+1)\rangle$
OG'10	1	$U^{\oplus 3} \oplus E_8^{\oplus 2} \oplus A_2$
OG'6	4	$U^{\oplus 3} \oplus (-2)^{\oplus 2}$

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$$\mathrm{Bir}(X) \longrightarrow \mathcal{O}(H^2(X,\mathbb{Z})) \text{ and } \mathcal{O}(H^2(X,\mathbb{Z})) \twoheadrightarrow \mathcal{O}(A_X)(\cong (\mathbb{Z}/2\mathbb{Z})^r)$$

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$\overset{[\text{Mon16}]}{\Longrightarrow}$ A finite subgroup $G \subset \mathcal{O}(H^2(X,\mathbb{Z}))$ is induced by a symplectic automorphism subgroup iff

- $(H^2(X,\mathbb{Z})^G)^\perp$ is non degenerate and negative definite;
- $(H^2(X,\mathbb{Z})^G)^\perp$ contains no numerical wall divisors;
- G acts trivially on A_X.

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Does there exist a birational involution ι such that $g = \iota \circ f$ with $\iota^* = R_e$ a reflection map on a class in cohomology, $\iota^*_{|A_X} = -1$ and $f^*_{|A_X} = 1$?

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Theorem

Let X be a projective hyperkähler manifold of $K3^{[n]}$–type admitting a symplectic birational map of finite order with a non–trivial action on A_X. Then, X is birational of a moduli space of (twisted) sheaves on a K3 surface.

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Theorem (DMP)

Suppose that S is a general K3 surface. Let $R_v \in \mathcal{O}(NS(M))$. There exists an involution $\iota_v \in Bir(M)$ symplectic with $\iota_v^* = -Id$ on A_X iff v satisfies

- $r|2c$, $\gcd(r, s) = 1$ or 2;

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$$\widetilde{H}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z}),$$

with the pairing

$$((r,l,s)\cdot(r',l',s')) = -rs' + (l\cdot l') - sr'.$$

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Let $\mathcal{F}$ be a stable sheaf on (S,H). The Mukai vector associated to $\mathcal{F}$ is given by

$$(\mathrm{rk}\mathcal{F}, c_1(\mathcal{F}), \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) + \mathrm{rk}\mathcal{F}).$$

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$$U(m) \hookrightarrow \widetilde{\Lambda}_{alg} \text{ for some } m \neq 0.$$

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- Set $\widetilde{\Lambda}$ be the extension of lattices and 2–weight Hodge structures of $H^2(X,\mathbb{Z})$ that is isometric to $U^4 \oplus E_8^2$. There exists a unique extension of g^* on $\widetilde{\Lambda}$ such that $g^*_{|A_X} = -1$:

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- $S_{g^*}(X) \oplus \langle v \rangle \subset S_{g^*}(\widetilde{\Lambda})$ and $\mathrm{sign} S_{g^*}(\widetilde{\Lambda})$ is $(1, r)$ where r is the rank of $S_{g^*}(X)$.

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$$A_{S_{g^*}(\widetilde{\Lambda})} \cong (\mathbb{Z}/2\mathbb{Z})^{\alpha_1} \oplus (\mathbb{Z}/2^2\mathbb{Z})^{\alpha_2} \oplus ... \oplus (\mathbb{Z}/2^k\mathbb{Z})^{\alpha_k}, \qquad (1)$$

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(sketch proof) Theorem I.

Note that $S_{g^*}(\widetilde{\Lambda})$ is hyperbolic but not always 2-elementary. Let us suppose that $\mathrm{ord}(g) = 2^k$. Since $\widetilde{\Lambda}$ is an unimodular lattice and $S_{g^*}(\widetilde{\Lambda})$ is a primitive sublattice of $\widetilde{\Lambda}$, the discriminant group $A_{S_{g^*}(\widetilde{\Lambda})}$ is isomorphic to $\widetilde{\Lambda}/(\widetilde{\Lambda}^{g^*} \oplus S_{g^*}(\widetilde{\Lambda}))$ which is 2^k–torsion, and so

$$A_{S_{g^*}(\widetilde{\Lambda})} \cong (\mathbb{Z}/2\mathbb{Z})^{\alpha_1} \oplus (\mathbb{Z}/2^2\mathbb{Z})^{\alpha_2} \oplus \dots \oplus (\mathbb{Z}/2^k\mathbb{Z})^{\alpha_k}, \tag{1}$$

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$\overset{Hass/Tsch}{\Longrightarrow}$: $Mov(M) = \bigcup_{\phi : M' \dashrightarrow M} \phi^*(Nef(M'))$.

Wall–crossing in the space of the stability conditions

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Theorem ([BM14])

Let v be a primitive Mukai vector.

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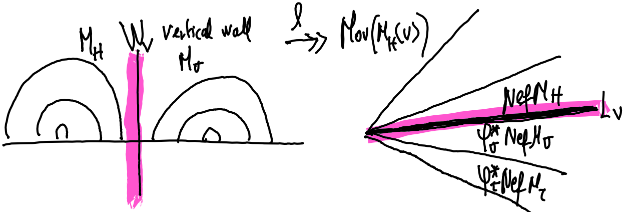

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Theorem ([BM14])

There exists birational contractions

$$\pi^\pm : M_{\sigma_\pm}(v) \to \overline{M}_\pm$$

where $\overline{M}_\pm$ are normal irreducible projective varieties. The curves contracted by $\pi^\pm$ are precisely the curves of objects that are S-equivalent to each other with respect to σ_0.

The type of birational transformation are described as follows.

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A wall $\mathcal{W}$ is called

1. a *fake wall* if there are no curves in $M_{\sigma_\pm}(v)$ of objects that are S-equivalent to each other with respect to σ_0,

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4. a *divisorial wall* if the morphisms $\pi^\pm : M_{\sigma_\pm}(v) \to \overline{M}_\pm$ are both divisorial contractions.

Assume $M \simeq M_\sigma(S, v)$ for a general K3 surface S, $\sigma \in Stab^+(S)$.

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Lemma

The reflection R_{e_v} belongs to $Mon^2(M)$ with $R_{e_v}|_{A_M} = -1$ if and only if

$$r \mid 2c \textit{ and } \gcd(r, s) = 1 \textit{ or } 2, \qquad (*)$$

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The reflection R_e belongs to $Mon^2(M)$ with $R_e\mid_{A_M} = -1$ if and only if

$$r \mid 2c \text{ and } \gcd(r, s) = 1 \text{ or } 2, \tag{$*$}$$

As a consequence,

- A potential wall is either divisorial or a flopping wall.

Proposition

Under the conditions $(*)$, the vertical wall is a divisorial wall

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Under the conditions (∗), the vertical wall is a divisorial wall if and only if one of the following cases occurs.

- $r = 1, r = 2$ *or* $c = kr$ *and* $s = (D^2/2)k^2 r - m$ *for some* $k \in \mathbb{Z}$ *and* $m = 1$ *or* 2.
- $r > 2$, $r \nmid c$, *and one of the two possibilities occurs.*
  1. $H^2 \equiv 0 \pmod 4$, $v = (2a, baD, \frac{D^2}{4}b^2 a - 1)$ *and* $w = (2, bD, \frac{D^2}{4}b^2)$ *for some integers* $a \geq 2$, b *odd.*
  2. $H^2 \equiv 2 \pmod 4$, $v = (2a, baD, \frac{D^2 b^2 a - 2}{4})$ *and* $w = (4, 2bD, \frac{D^2}{2}b^2)$ *for some integers* $a \geq 2$ *odd,* b *odd.*

Examples

Let S be a K3 surface of $\mathrm{Pic}(S) = \langle H \rangle$. Set r, $s \in \mathbb{Z}$, $\gcd(r, s) = 1$:

- r $= 1$: In this case, $X = M_H(1, 0, -s) = S^{[1+s]}$ and $e = [E]/2$ where $E \subset S^{[1+s]}$ is the diagonal, so E corresponds to the prime divisor which is the exceptional locus of the Hilbert–Chow morphism $\epsilon : S^{[1+s]} \longrightarrow S^{(1+s)}$.

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