Moduli space for n points on a projective line

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Spring school Invariants in Algebraic Geometry, Dijon 2022

Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

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Der Wissenschaftsfonds.

[1]The author is supported by the Austrian Science Fund (FWF): W1214-N15, project DK9.

motivation

- The compactification we will discuss is denoted by M_n in this talk. It is a smooth projective variety of dimension $n - 3$. It has been constructed by Knudsen and Mumford.
- Their construction has been used for theoretical physics, resolution of singularities, and kinematics. It has been called "the main tool of modern enumerative geomety".
- However, their construction is very long and complicated. We will give a self-contained construction of a variety which is isomorphic to the Knudsen-Mumford moduli space, using only basic algebraic geometry.
- We will not go into details of their construction.

cross ratio

- Given a quadruple $(p_1, p_2, p_3, p_4) \in (\mathbb{P}^1)^4$.
- If the four points are pairwise distinct, it's **cross ratio** is defined to be $((p_1 - p_3)(p_2 - p_4) : (p_1 - p_4)(p_2 - p_3))$.
- Later we use the notation $\gamma_q(p)$, where $p \in (\mathbb{P}^1)^n$ and q a quadruple of four entries, to define the cross ratio of these four entries on p.
- It is normally extended to the case when one of the entries are infinity; basically just remove the corresponding two differences from the formula.

cross ratio

- We introduce the abbreviations ∞, $\mathbf{0}$, $\mathbf{1}$ for the three points $(1:0),(0:1),(1:1) \in \mathbb{P}^1$, respectively.
- When the four places are pairwise distinct, it's not hard to check that the cross ratio is then different from ∞, $\mathbf{0}$, or $\mathbf{1}$. In other cases, the definition is the following:
- $p_1 = p_2$ or $p_3 = p_4$ iff $\gamma(p_1, p_2, p_3, p_4) = \mathbf{1}$;
  $p_1 = p_3$ or $p_2 = p_4$ iff $\gamma(p_1, p_2, p_3, p_4) = \mathbf{0}$;
  $p_1 = p_4$ or $p_2 = p_3$ iff $\gamma(p_1, p_2, p_3, p_4) = \infty$.
- If three or four places coincide in the quadruple, we say that the cross ratio **is not defined**.
- When this definition is clear, we can then move forward to the basic settings.

basic settings

- Let $n \geq 3$ be an integer, we study the equivalence induced by the group action of $PGL(2, \mathbb{C})$ on $(\mathbb{P}^1)^n$. We can also view it as a Möbius transformation applied on each entry of the sequence. (Elements in $PGL(2, \mathbb{C})$ are all the 2×2 matrices which has non-zero determinant.)
- Two n-tuples are equivalent if there is a projective linear transformation transforming one into the other.
- In our setting this transformation is nothing more than Möbius transformation.
- A Möbius transformation of the complex plane is a rational function of the form $f(z) = \frac{az+b}{cz+d}$ of one complex variable z; a, b, c, d here are complex numbers satisfying $ad - bc \neq 0$.

basic settings

- When the n-tuples have n distinct points, two n-tuples are equivalent if and only if all cross ratios defined by all (corresponding) quadruples coincide.
- In this case, the equivalence classes are in bijective correspondence with the points of an open subset $(\mathbb{P}^1)^{n-3}$, which can be parametrized by $n-3$ cross ratios. (Because of the 3-sharp-transitivity of PGL_2, we can fix three coordinates.)
- 3-sharp-transitivity: there is a unique group element which transfers the three pairwise distinct points to another three pairwise distinct points.

notations

- $N := \{1, \ldots, n\}$, where $n \geq 3$ is a natural number. Elements of it are called **nodes**.
- An n-tuple $(p_1, \ldots, p_n) \in (\mathbb{P}^1)^n$ is called an **n-gon**.
- An n-gon is **dromedary** if all its places are distinct. p_is are called **places** of the n-gon.
- PGL_2 acts on $(p_1, \ldots, p_n)$ by $(p_1, \ldots, p_n)^\sigma := (p_1^\sigma, \ldots, p_n^\sigma)$ for all $\sigma \in PGL_2$. The equivalent classes are called **orbits**.
- Dromedary orbits (orbits of dromedary n-gons) are in bijective correspondence with the points in U_n.
- U_n is defined as the open subset of all points $(c_4, \ldots, c_n) \in (\mathbb{P}^1)^{n-3}$ where $c_i \notin \{\infty, \mathbf{0}, \mathbf{1}\}$ for $i \in \{4, \ldots, n\}$ and $c_i \neq c_j$ if $i \neq j$, where $i, j \in \{4, \ldots, n\}$. (When we transfer n distinct points on $\mathbb{P}^1$, after the transformation, they stay pairwise distinct.)

notations

- U_n is the moduli space of n distinct points on $\mathbb{P}^1$, under PGL_2 group action.
- It is an open subset of $(\mathbb{P}^1)^{n-3}$, and $(\mathbb{P}^1)^{n-3}$ is indeed a compactification of it, which is projective and smooth. However, the first three entries are somehow special, so it is not symmetric under random permutation of the nodes.
- We want to find a good compactification of U_n which is smooth, symmetric under permutation of nodes, projective.
- Basically we need to consider those orbits that are not dromedary, and make a compactification of U_n out of them.
- We manage to find it! It is denoted by M_n in this talk.

moduli space

- Denote by $T_n := \{(i,j,k) \mid i,j,k \in \{1,...,n\}, i<j<k\}$.
- Sometimes we use short notation for the elements in T_n, for instance, 123 represents $\{1,2,3\}$,etc.
- $M_n := \{p \in ((\mathbb{P}^1)^n)^{T_n} \mid \forall t=(i,j,k) \in T_n : p_i^t = \infty, p_j^t = \mathbf{0}, p_k^t = \mathbf{1}, \forall t_1, t_2 \in T_n, \forall q \in Q : \gamma_q(p^{t_1}) = \gamma_q(p^{t_2})$ if both sides are defined$\}$.
- Note that we define M_n only for $n \geq 3$, otherwise there is no triple to consider..
- Let's see some examples, so as to better understand the definition.
- When $n=3$, M_3 consists of only one element which can be denoted as p. p contains only one 3-gon: $p^{(1,2,3)}$. We have $p_1^{(1,2,3)} = \infty$, $p_2^{(1,2,3)} = \mathbf{0}$, $p_3^{(1,2,3)} = \mathbf{1}$.
- Since the number of entries is not enough to talk about cross ratios, with this we finish the exploration of M_3.

moduli space: examples (M_3)

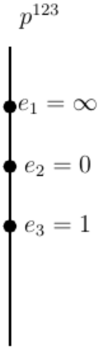


Figure: Here is the graphical representation of the unique element in M_3, inside which the vertical line segment represents $\mathbb{P}^1$.

moduli space: examples (M_4)

- When $n = 4$. M_4 consists of infinitely many elements. Each one of them contains four elements: p^{123}, p^{124}, p^{134}, p^{234}. Denote any element in M_4 by p.
- **When four entries of p are pairwise distinct**, we have that $p_1^{123} = \infty$, $p_2^{123} = \mathbf{0}$, $p_3^{123} = \mathbf{1}$, assume w.l.o.g., $p_4^{123} = a$, where $a \in \mathbb{P}^1 \setminus \{\infty, \mathbf{0}, \mathbf{1}\}$.
- Actually, after a simple computation we see that $\gamma_{(1,2,3,4)}(p^{123}) = p_4^{123} = a$.
- With the requirement on cross ratios in the definition of M_n, we can calculate out precisely the other three 4-gons.
- Since $\gamma_{1234}(p^{123}) = \gamma_{1234}(p^{124})$, we can get that $p_3^{124} = \frac{1}{a}$. Analogously, we obtain that $p_2^{134} = \frac{1}{1-a}$ and $p_1^{234} = \frac{a}{a-1}$.

moduli space: examples (M_4)

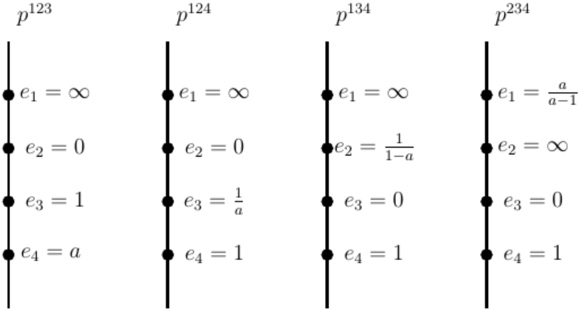


Figure: Here is the graphical representation of an arbitrary element in M_4, of which all four entries are pairwise distinct. $\gamma_{1234}(p) = a$. Note that here if we apply a PGL_2 group action to the 4-gons of this element p, we obtain only one orbit, the structure of which is a 4-gon with four pairwise distinct entries.

moduli space: examples (M_4)

- Since we only discuss here the situation when $n \geq 3$, there should be at least three entries. So the only case that is left is **when two entries coincide**.
- There are in total three elements in M_4 in this case.
- First one is $p_1^{123} = p_4^{123}$. Then by the requirement of cross ratio in the definition, we deduce that $p_2^{124} = p_3^{124}$, $p_2^{134} = p_3^{134}$ and $p_4^{234} = p_1^{234}$.
- Second one is $e_2 = e_4$ on p^{123} and p^{134}, $e_1 = e_3$ on p^{124} and p^{234}.
- Third one is $e_3 = e_4$ on p^{123} and p^{124}, $e_1 = e_2$ on p^{134} and p^{234}.
- We will show the first one in a graphical way on the next slide.

moduli space: examples (M_4)

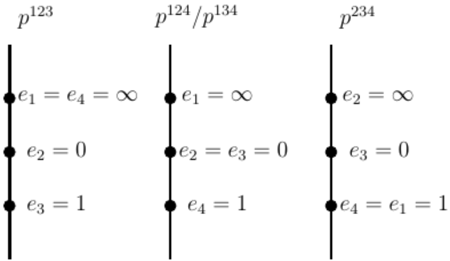


Figure: Here is the graphical representation of an element which has two entries coincide in M_4. $\gamma_{1234}(p) = \infty$. Note that here if we apply PGL_2 group action to the 4-gons of this element in M_4, we obtain two distinct orbits. One of which has $e_1 = e_4$ and the other has $e_2 = e_3$.

loaded graph

- Let $x \in M_n$. (so x is a set of n-gons fulfilling the cross ratio condition)
- If p is an n-gon of x, then a subset $I \subset N$ is called a **cluster** of p or of its orbit (under PGL_2 action) $[p]$, iff $\forall i, j \in I, k \in N \setminus I$ we have $p_i = p_j \neq p_k$.
- A cluster I is **proper** if and only if it has at least two elements.
- For each $x \in M_n$, we define a graph (V, E) as the following.
- V is the set of all PGL_2-orbits of n-gons of x.
- There is an edge between $[p]$ and $[q]$ iff $[p]$ has a cluster I, $[q]$ has a cluster J and (I, J) is a bi-partition of N.
- For each vertex v, $H(v)$ is the set of nodes i such that $\{i\}$ is a cluster of v. We call it the **singletons** of v.

loaded graph

- The graph (V, E), together with the subsets $H(v)$ for $v \in V$, is called the **loaded graph** of x and denoted by $L(x)$.
- If $x \in U_n$, then all its n-gons are PGL_2-equivalent. Hence $L(x)$ has only a single vertex v. There are no proper clusters, hence also no edges in $L(x)$. Every node is a singleton, hence $H(v) = N$.
- Let's see some examples.

loaded graph: examples-recall

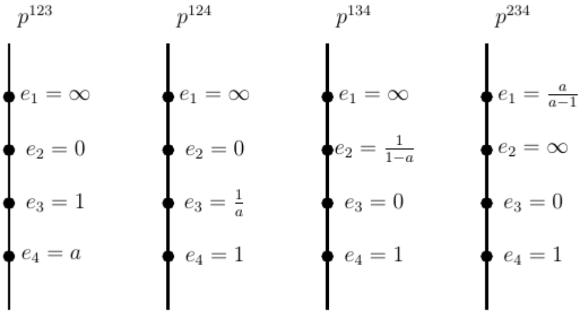


Figure: Here is the graphical representation of an arbitrary element in M_4, of which all four entries are pairwise distinct. $\gamma_{1234}(p) = a$.

loaded graph: examples

- For the above element in M_4, we get only one orbit under the PGL_2 group action. Therefore, in the loaded graph, there is only one vertex v.
- $H(v) = \{1, 2, 3, 4\}$.
- Graphically, we can view it as the following.

loaded graph: examples-recall

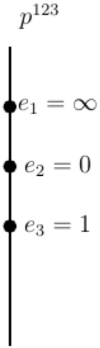
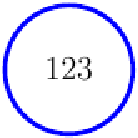


Figure: Here is the graphical representation of the unique element in M_3, inside which the vertical line segment represents $\mathbb{P}^1$.

loaded graph: examples

- For that unique element in M_3, there is only one orbit under PGL_2 group action. Hence there is only one vertex for the loaded graph.
- Singletons of v are $\{1, 2, 3\}$, we can view it graphically as the following:



loaded graph: examples-recall

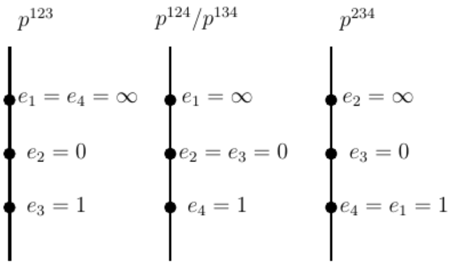


Figure: Here is the graphical representation of an element which has two entries coincide in M_4. $\gamma_{1234}(p) = \infty$.

loaded graph: examples

- If we consider the PGL_2 group action on this element in M_4, there are two orbits: one with $e_1 = e_4$ and pairwise distinct with e_2, e_3; the other with $e_2 = e_3$ and pairwise distinct with e_1, e_4.
- To view it graphically, see the next slide.

loaded graph: examples

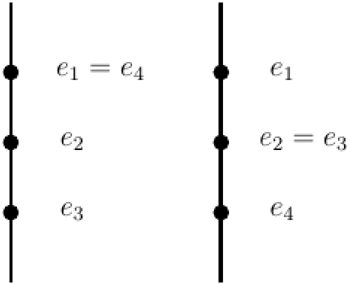


Figure: Two orbits of an element in M_4 where two entries coincide, under PGL_2 group action.

loaded graph: examples

- Continue with this element, there are two vertices in its loaded graph, v_1 and v_2. $H(v_1) = \{2, 3\}$, $H(v_2) = \{1, 4\}$.
- How about edges?
- Since orbit v_1 has a cluster $\{1, 4\}$, v_2 has a cluster $\{2, 3\}$, they together is a bi-partition of $\{1, 2, 3, 4\}$. So there is an edge between v_1 and v_2.
- We see this graph in the following:

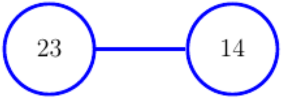


Figure: Note that here the vertex on the left represents v_1 and that on the right represents v_2.

loaded graph: properties

let $x \in M_n$.

Lemma

A cluster $I \subset N$ cannot be a cluster of two distinct orbits of x.

Lemma

If J is a proper cluster of x, then $N \setminus J$ is also a (proper) cluster of x.

Remark

From the above two lemmas, we know that for any proper cluster of v, there is a unique edge corresponding to it in the loaded graph (where v is one of its vertices).

loaded graph: properties

Lemma

Every node $i \in N$ is a singleton of exactly one orbit of n-gons.

Remark

Non-empty sets $H(v)$ form a partition of N.

loaded graph: properties

Lemma

For every orbit v*, we have* $|H(v)| + \deg(v) \geq 3$*, where* $\deg(v)$ *is the vertex degree with respect to the loaded graph* (V, E)*.*

Remark

Every orbit must have at leat three distinct places, by definition.

loaded tree

Lemma

For any $x \in M_n$, the loaded graph of x is a tree.

- proof sketch:
- First we show by a proper inclusion of clusters that there is no cycle in the graph.
- Then we show by induction that for any two vertices u, v, there is a path in (V, E) connecting them.

loaded tree

A "loaded tree with node set N" is a tree (V, E) together with a collection $(H(v))_{v \in V}$ of subsets of N so that its non-empty elements form a partition of N, and $|H(v)| + \deg(v) \geq 3$ holds for each $v \in V$.

Theorem

Let (V, G, H) be the loaded graph of $x \in M_n$. Then (V, G, H) is a loaded tree with the node set N.

The converse statement also holds.

Theorem

Let (V, G, H) be a loaded tree with the node set N. Then there exists a point $x \in M_n$ such that $L(x) = (V, G, H)$.

We denote loaded tree of $x \in M_n$ as $LT(x)$.

loaded tree: application

- Here we want to apply the second theorem on last page, trying to find all loaded trees with the node set $N = \{1, 2, 3, 4, 5\}$.
- Let us try it on the blackboard!
- Note that loaded trees is just one way of grouping the elements in M_n. One loaded tree can represent infinitely many different elements; however, sometimes can also just represent one element.

smoothness

With the help of its combinatorics structures, we can prove the following result.

Theorem

The variety M_n is smooth and of dimension $n - 3$.

- So indeed, it is a compactification of U_n which is projective, smooth, and symmetric with respect to the nodes.
- Furthermore, we managed to prove that it is isomorphic to the Knudsen-Mumford moduli space.

Thank You