Tame degree functions in arbitrary characteristic

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$\deg : B \to G \cup \{-\infty\}$: a degree function fixed throughout.

For integral domains $C \subset A$,
$A = C^{[n]}$ denotes: $A = C[t_1, \ldots, t_n]$ for elements $t_1, \ldots, t_n \in A$ algebraically independent over C.

Motivation: D. Daigle's work

Definition

A function $\deg : B \to G \cup \{-\infty\}$ is called a *degree function* if it satisfies, for all $a, b \in B$:

(i) $\deg(a) = -\infty$ if and only if $a = 0$.

(ii) $\deg(ab) = \deg(a) + \deg(b)$.

(iii) $\deg(a + b) \leq \max\{\deg(a), \deg(b)\}$.

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Definition

For each $i \in G$, let $B_i = \{x \in B \mid \deg(x) \leq i\}$, $B_{i^-} = \{x \in B \mid \deg(x) < i\}$. Then $\mathrm{Gr}(B) = \bigoplus_{i \in G}(B_i / B_{i-})$ is the *associated graded ring of B induced by the degree function* $\deg$ on B.

Motivation: D. Daigle's work

- Given a derivation $D : B \to B$,

  $\phi \neq U = \{\deg(D(x)) - \deg(x) \mid x \in B \setminus \{0\}\} \subset G \cup \{-\infty\}.$

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- $\deg(D) := \sup U$ (if it exists), otherwise we say that $\deg(D)$ is not defined.
- (Homogenization of derivations) Each derivation $D : B \to B$ such that $\deg(D)$ is defined gives rise to a homogeneous derivation $\mathrm{gr}(D) : \mathrm{Gr}(B) \to \mathrm{Gr}(B)$.

Definition

Let $A \subseteq B$ be integral domains of characteristic zero. A degree function $\deg : B \to G \cup \{-\infty\}$ is said to be *tame over A* or *A-tame*, if $\deg(D)$ is defined for all A-derivations $D : B \to B$. If deg is not tame over A, we say that it is *wild over A*.

Motivation: D. Daigle's work

- Confusion regarding degree functions:
  Consider the following incorrect statement
  "If B is a domain and a finitely generated $\mathbb{C}$-algebra, then all degree functions on B are tame over $\mathbb{C}$." $(\times)$

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- However this false assertion has been used several times to justify the homogenization of derivations by notable mathematicians!

- In fact, there are many examples in the literature where author simply omits to raise the question if $\deg(D)$ is defined, as if it were a priori clear that $\deg(D)$ is always defined!

Why this work?

Replacing a derivation by a suitable homogenization constructed out of a degree function, has turned out to be very useful in Affine Algebraic Geometry.

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- It was the main technical tool of Makar-Limanov to show that the Russell-Koras threefold over $\mathbb{C}$ is not a polynomial ring.
- It was the central tool of N. Gupta to show that most Asanuma threefolds are not polynomial rings, thus providing counter-examples to the Zariski Cancellation Problem for $k^{[3]}$, when k is a field of positive characteristic.

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It is therefore imperative to have results similar to Daigle over an integral domain B containing a field k of arbitrary characteristic.

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- To formulate the definition of the tameness of a degree function in arbitrary characteristic.
- To establish some results regarding extension of degree function under localization.
- To prove sufficient conditions for tameness of degree functions.

Locally Finite (Iterative) Higher Derivation

Definition 1

A *locally finite higher derivation (lfhd)* on a k-algebra B is a sequence of k-linear endomorphisms $D = \{\phi_0, \phi_1, \dots\}$ of the k-vector space B satisfying the following properties:

(i) $\phi_0 = \mathrm{Id}_B$,

(ii) $\phi_i(ab) = \sum_{j+\ell=i} \phi_j(a)\phi_\ell(b)$ for all $a, b \in B$,

(iii) For each $b \in B$ there exists $n \in \mathbb{N}$ such that $\phi_m(b) = 0$ for all $m \geq n$.

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Definition 2

A lfhd $D = \{\phi_0, \phi_1, \dots\}$ on a k-algebra B is said to be *iterative* if $\phi_i\phi_j = \binom{i+j}{i}\phi_{i+j}$ for all $i, j \geq 0$. Such a derivation is also known as *exponential map* on B.

Locally Finite (Iterative) Higher Derivation

Remark 1

For any lfhd $D = \{\phi_0, \phi_1, \dots\}$ on B, the k-algebra map $\phi : B \to B[T] = B^{[1]}$ given by $\phi = \sum_{i \geq 0} \phi_i T^i$ is a k-algebra homomorphism. Note that $\varepsilon_0 \phi = \mathrm{Id}_B$, where $\varepsilon_0 : B[T] \to B$ is the evaluation map at $T = 0$. Conversely, any k-algebra homomorphism $\phi : B \to B[T] = B^{[1]}$ satisfying $\varepsilon_0 \phi = \mathrm{Id}_B$ gives rise to a lfhd $D = \{\phi_i\}_{i \geq 0}$ on B.

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Remark 2

When k is a field of characteristic zero, then ϕ_1 defines a locally nilpotent derivation on B and $\phi_n = \frac{1}{n!} {\phi_1}^n$. However, when the characteristic of k is positive, then ϕ_1 alone does not determine the entire sequence $D = \{\phi_i\}_{i \geq 0}$.

Degree of a LFHD

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Definition 3

Let $\delta_\phi : B \to F(G) \cup \{-\infty\}$ denote the function defined by

$$\delta_\phi(x) = \begin{cases} \sup_{i>0} \left\{ \frac{\deg(\phi_i(x)) - \deg(x)}{i} \right\} & \textit{for } x \neq 0 \\ -\infty & \textit{for } x = 0. \end{cases}$$

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Definition 4

For any subset $U \subseteq B$, let

$$\delta_\phi(U) = \sup_{x \in U} \{\delta_\phi(x)\} \textit{ if it exists};$$

otherwise we let $\delta_\phi(U)$ to be undefined. Define $\delta_\phi(\varnothing) = -\infty$.

Degree of a LFHD

Definition 5

If $\sup_{i>0}\left\{\frac{\deg(\phi_i(x))-\deg(x)}{i}\middle| x\in B\right\}$ exists, then we set this to be the degree of ϕ and denote it by $\deg(\phi)$.
Thus,

$$\deg(\phi)=\sup_{x\in B}\{\delta_\phi(x)\}=\delta_\phi(B).$$

Tameness of a degree function

Definition 6

Let $A \subseteq B$ be integral domains. A degree function $\deg : B \to G \cup \{-\infty\}$ is said to be *tame over A* or *A-tame*, if $\deg(\phi)$ is defined for every k-algebra homomorphism $\phi : B \to B[T]$ satisfying

(i) $\varepsilon_0 \phi = \mathrm{Id}_B$ (i.e., ϕ is associated with a locally finite higher derivation) and

(ii) $A \subseteq B^{\phi}$.

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If deg is not tame over A, we say that it is *wild over A* or *A-wild*.

Extension of 'deg' under localization

Let $\phi : B \to B[T]$: a k-algebra homomorphism associated with a lfhd $D = \{\phi_i\}_{i\geq 0}$ & $S \subseteq B \setminus \{0\}$: a m.c. subset. Then ϕ extends to a homomorphism $\Phi : S^{-1}B \to S^{-1}B[[T]]$ given by

$$\Phi(b/s) = \frac{\phi(b)}{\phi(s)} \text{ for } b \in B, s \in S.$$

The homomorphism $\Phi = \sum_{i\geq 0} \Phi_i T^i$ corresponds to a higher derivation $\bar{D} = \{\Phi_i\}_{i\geq 0}$ on $S^{-1}B$ such that $\Phi_i|_B = \phi_i$ for each $i \geq 0$.

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Now $\deg : B \to G \cup \{-\infty\}$ has a unique extension

$$\text{DEG} : S^{-1}B \to G \cup \{-\infty\},$$

where

$$\text{DEG}(b/s) = \begin{cases} \deg(b) - \deg(s) \ \textit{for } 0 \neq b \in B, s \in S \\ -\infty \ \textit{for } b = 0. \end{cases}$$

Extension of 'deg' under localization

Lemma 1

For $b \in B \setminus \{0\}$ and $s \in S$,

$$\delta_\Phi(b/s) := \sup_{i>0} \left\{ \frac{\mathrm{DEG}(\Phi_i(b/s)) - \mathrm{DEG}(b/s)}{i} \right\}$$

$$\leq \max\{\delta_\phi(b), \delta_\phi(s)\}$$

Thus $\delta_\Phi : S^{-1}B \to F(G) \cup \{-\infty\}$ is a well-defined map.

Extension of ‘deg’ under localization

Definition 7

With notation as above, we define the *degree of* Φ to be

$$\mathrm{DEG}(\Phi) = \sup_{x \in S^{-1}B} \{\delta_\Phi(x)\} = \delta_\Phi(S^{-1}B) \quad \text{(if it exists).}$$

Extension of ‘deg’ under localization

Definition 7

With notation as above, we define the *degree of* Φ to be

$$\mathrm{DEG}(\Phi) = \sup_{x \in S^{-1}B} \{\delta_{\Phi}(x)\} = \delta_{\Phi}(S^{-1}B) \ \text{ (if it exists).}$$

Lemma 2

With the notation as above, $\deg(\phi)$ is defined if and only if $\mathrm{DEG}(\Phi)$ is defined. Also, if both the degrees are defined then they are equal.

Filtration

Definition 8

Let $\mathcal{F} = \{B_i\}_{i \in G}$ be a collection of k-linear subspaces of B and let $B_{i^-} := \{x \in B \mid x \in B_j \text{ for some } j < i\}$. Then $\mathcal{F} = \{B_i\}_{i \in G}$ is said to define a *proper G-filtration* if it satisfies the following conditions:

(i) $B_i \subseteq B_j$ for all $i \leq j$; $i, j \in G$,

(ii) $B = \bigcup_{i \in G} B_i$,

(iii) $\bigcap_{i \in G} B_i = \{0\}$ and

(iv) $(B_i \setminus B_{i^-}).(B_j \setminus B_{j^-}) \subseteq B_{i+j} \setminus B_{(i+j)^-}$ for all $i, j \in G$.

A proper G-filtration $\mathcal{F} = \{B_i\}_{i \in G}$ of B is called *admissible* if there exists a finite generating set Γ of B such that for each $i \in G$ and $b \in B_i$, b can be written as a finite sum of monomials in elements of Γ and each of these monomials is an element of B_i.

Filtration & degree function

Remark 3

Let $\mathcal{F} = \{B_i\}_{i \in G}$ be a proper G-filtration on B. Then

$$\deg_{\mathcal{F}}(x) := \begin{cases} i & \textit{if } x \in B_i \setminus B_{i^-} \\ -\infty & \text{otherwise} \end{cases}$$

defines a degree function $\deg_{\mathcal{F}} : B \to G \cup \{-\infty\}$ on B. Conversely, any degree function $\deg : B \to G \cup \{-\infty\}$ defines a filtration on B by setting

$$B_i := \{x \in B \mid \deg(x) \leq i\}.$$

Sufficient conditions for tameness of 'deg'

Theorem 1

Let B be an affine k-domain with an admissible proper G-filtration $\mathcal{F} = \{B_i\}_{i \in G}$ and $\deg : B \to G \cup \{-\infty\}$ the degree function induced by the filtration. Then deg is tame over k.

In fact, if $z_1, \ldots, z_m \in B$ be such that $\Gamma = \{z_1, \ldots, z_m\}$ is a generating set making the filtration admissible, then

$$\deg(\phi) = max\{\delta_\phi(z_1), \ldots, \delta_\phi(z_m)\}$$

for all k-algebra homomorphism $\phi : B \to B[T]$ associated with a lfhd $D = \{\phi_i\}_{i \geq 0}$.

Sufficient conditions for tameness of 'deg'

Theorem 2

Let $S \subseteq B \setminus \{0\}$ be a m.c. subset such that $S^{-1}B = \bigoplus_{i \in G} F_i$ is a G-graded domain. Suppose that $S^{-1}B$ is a finitely generated F_0-algebra and $\operatorname{tr.deg}_A F_0 < \infty$, for some subring $A \subseteq B \cap F_0$. Let $\deg : B \to G \cup \{-\infty\}$ be the degree function of B induced by the grading of $S^{-1}B$. Then deg is tame over A.

More specifically, let $z_1, \ldots, z_m \in F_0$ be such that F_0 is algebraic over $A[z_1, \ldots, z_m]$ and let $x_1, \ldots, x_n \in S^{-1}B$ be homogeneous elements such that $S^{-1}B = F_0[x_1, \ldots, x_n]$. Then

$$\deg(\phi) \leq \max\{\delta_\Phi(z_1), \ldots, \delta_\Phi(z_m), \delta_\Phi(x_1), \ldots, \delta_\Phi(x_n)\}$$

for all k-algebra homomorphism $\phi = \{\phi_i\}_{i \geq 0} : B \to B[T]$ associated with a lfhd of B satisfying $A \subseteq B^\phi$, where $\Phi : S^{-1}B \to S^{-1}B[[T]]$ is its extension.

Sufficient conditions for tameness of 'deg'

Theorem 3

Let $B = \bigoplus_{i \in G} B_i$ be a G-graded k-domain and $\deg : B \to G \cup \{-\infty\}$ the degree function determined by the grading. Suppose that B is f.g. as a B_0-algebra and A be a subring of B_0 such that $\operatorname{tr.deg}_A(B_0) < \infty$. Then deg is tame over A.

References

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