Th W is generic, quasi-symmetric + technical condition then

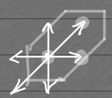
$\mathcal{L} = \left(-\bar{\rho} + \frac{1}{2}\bar{\Sigma} + \varepsilon\right) \cap X(T)^+$ for $\varepsilon \in X(G)_{\mathbb{R}}$ generic

$U = \bigoplus_{\chi \in \mathcal{L}} V(\chi)$

$\Lambda = \mathrm{End}_R(M(U))$ is an NCCR of $R = k[W]^G$.

Ex¹. $U = \bigoplus_{i=-2}^{2} V(i)$, $\Lambda = \mathrm{End}_R(M(U))$ is an NCCR of R.

Ex². $U = V(0,0) \oplus V(1,0) \oplus V(1,1)$, $\Lambda = \mathrm{End}_R(M(U))$ is an NCCR of R.



Ex.
- determinantal varieties [Buchweitz, Leuschke, VdB 2010]
- Pfaffian varieties $Z_{n,2r}$ if n odd
- determinantal varieties for symmetric matrices $Z^+_{n,r}$ if $n \equiv r+1 \pmod 2$
- (commutative) trace rings (twisted NCCRs)

- [Broomhead 2012] Gorenstein normal affine toric varieties of dim 3

cEx. [Quarles 2005]

$G = SL_2 = SL(V)$, $\dim V = 2$, $W = V^{\oplus 4}$

$k[W]^G = k[x_1, \dots, x_6]/(x_1^2 + \dots + x_6^2)$ does not have an NCCR

3.5 Idea of the proof

- If $\mathrm{pdim}\, \mathrm{Hom}_R(M(U), V(\chi)) < \infty$ $\forall \chi \in X(T)^+$, then $\mathrm{gldim}\, \Lambda < \infty$
- $Q_\chi := \mathrm{Hom}_R(M(U), V(\chi)) = \left(k[W] \otimes \mathrm{Hom}(U, V(\chi))\right)^G$
- Relate Q_χ using equivariant Koszul resolutions of $V(\chi) \otimes k[W/W^{\lambda>0}]$ where $W^{\lambda>0} = \{w \mid \lim_{t\to 0} \lambda(t)w = 0\}$

(e.g. $W = \mathbb{C}^4 \circlearrowleft \mathbb{C}^*$, $-1, -1, 1, 1$: $W^{>0} = \{0\} \times \{0\} \times \mathbb{C}^2$, $W^{<0} = \mathbb{C}^2 \times \{0\} \times \{0\}$)

4. COMMUTATIVE VS NONCOMMUTATIVE RESOLUTIONS

4.1 TILTING OBJECTS

Def. Y smooth, $\mathcal{T} \in D^b(Y)$
$\mathcal{T}$ is tilting if
- $\mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{T}, \mathcal{T}) = 0 \quad \forall i \in \mathbb{Z}$
- $\mathrm{thick}(\mathcal{T}) = D^b(Y)$

Ex $\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(1)$ is tilting on $\mathbb{P}^1$.
$\mathcal{T} = \mathcal{O} \oplus \mathcal{O}(1) \oplus \dots \oplus \mathcal{O}(n)$ is tilting on $\mathbb{P}^n$

Thm. $Y \to \mathrm{Spec}\, R$ projective, $\mathcal{T}$ tilting on Y, $\Lambda = \mathrm{End}_{\mathcal{O}_X}(\mathcal{T})$

Then
$$\begin{aligned} D^b(\mathrm{coh}\, Y) &\longrightarrow D^b(\Lambda) \\ \mathcal{F} &\longmapsto R\mathrm{Hom}_Y(\mathcal{T}, \mathcal{F}) \end{aligned}$$
is an equivalence of Δ-categories.

4.2 DIMENSION 3

Th [VdB, 2004]
$X = \mathrm{Spec}\, R$, $\dim X = 3$, X has term. sing.
If $\pi : \tilde{X} \to X$ is a crepant resolution of singularities, then there exists a tilting object $\mathcal{T} \in D^b(\tilde{X})$. In particular, $\mathrm{End}_{\mathcal{O}_X}(\mathcal{T})$ is an NCCR of R.

If Λ is an NCCR of R, then there exists $\pi : \tilde{X} \to X$ a crepant resolution of singularities. $\tilde{X}$ is obtained as a moduli space of certain stable representations of Λ.

Cor $X = \mathrm{Spec}\, R$, $\dim X = 3$, X has term. sing.
$\exists$ CCR of $X \iff \exists$ NCCR of R
All CCRs and NCCRs are derived equivalent.

4.3 DETERMINANTAL VARIETIES

$X_{m,r} = \{ \varphi \in M_m(\mathbb{C}) \mid \mathrm{rk}\, \varphi \leq r \} \subseteq M_m(\mathbb{C})$

Fix vec.sp. Q, $\dim Q = r$.

$\varphi \in X_{m,r}$
$$\begin{tikzcd} \mathbb{C}^m \arrow[rr, "\varphi"] \arrow[dr, "\varphi_1"'] & & \mathbb{C}^m \\ & Q \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

$(\varphi_1 : \mathbb{C}^m \twoheadrightarrow Q) \in \mathrm{Gr}(r, m)$

$$\begin{tikzcd} Z = \{(\varphi_1,\varphi_2) \mid \varphi_1 \in Gr(r,m),\ \varphi_2 : Q \to \mathbb{C}^n\} \arrow[r, "(\varphi_1,\varphi_2)\mapsto \varphi_2\varphi_1", "\text{birational, crepant}"'] \arrow[d, "pr_1"', "\text{vec. bundle}"] & X_{r,m} \\ Gr(r,m) \end{tikzcd}$$

Thm [Kapranov 1984]
$Gr(r,m)$ has a tilting bundle $\mathcal{T} = \bigoplus_{\alpha \in \square_{m-r}^{r}} S^\alpha Q$, Q is universal quotient bundle on $Gr(r,m)$)

Thm [Buchweitz, Leuschke, Van den Bergh 2010]
$pr_1^* \mathcal{T}$ is a tilting bundle on Z.
$\Lambda = \mathrm{End}_{\mathcal{O}_Z}(pr_1^*\mathcal{T})$ is an NCCR of $k[Y_{m,r}]$.

Rmk. Λ coincides with the NCCR that we would get from the construction from Π.

Rmk. For Pfaffian varieties $\exists\, Z' \to Y_{n,2r}$ similar to Z but it is not crepant, CCR does not exist, NCCR Λ does if n odd.
However, $D^b(\Lambda) \hookrightarrow D^b(Z)$ ("minimality of cr. res.")

4.4 VARIATION OF GIT

Relation between $X^{ss,\chi}/\!\!/G$ for $\chi \in X(G)$.

If $G = k^* \curvearrowright W$, $X(k^*) \cong \mathbb{Z}$. For $m \in \mathbb{N}_{>0}$. $\underset{X^{ss,+}}{\overset{\parallel}{X^{ss,m}}} \neq X^{ss,0} \neq \underset{X^{ss,-}}{\overset{\parallel}{X^{ss,-m}}}$.

E.g. $W = k^4$, with weights $-1,-1,1,1$

Then
$$\begin{tikzcd} W^{ss,+}/\!\!/G \cong Bl_1 W \arrow[rr, dashed, "\text{flop}"'] \arrow[dr] & & W^{ss,-}/\!\!/G \cong Bl_{r-1} W \arrow[dl] \\ & W/\!\!/G & \end{tikzcd}$$

but
$$\begin{tikzcd} D^b(W^{ss,+}/\!\!/G) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\wr\!\!\parallel"] & D^b(W^{ss,-}/\!\!/G) \arrow[dl, phantom, "\cong"] \\ D^b\Big(\underbrace{\mathrm{End}_R\begin{pmatrix} R & I^{-1} \\ I & R \end{pmatrix}}_{\text{NCCR}}\Big) & \end{tikzcd}$$

Thm [Halpern-Leistner, Sam 2016]
$\chi_1, \chi_2 \in X(G)$ "generic"

Then
$$\begin{tikzcd} D^b([X^{ss,\chi_1}/G]) \arrow[r, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & D^b([X^{ss,\chi_2}/G]) \arrow[d, phantom, "\cong"] \\ & D^b(\Lambda) \end{tikzcd}$$
($[X^{ss,\chi_i}/G]$ DM stacks)

Λ NCCR, obtained from the recipe above.
Moreover, $\exists$ tilt. bundle on $[X^{ss,\chi}/G]$ $\quad \bigoplus_{\mu \in \mathcal{L}} V(\mu) \otimes \mathcal{O}_{X^{ss,\chi}}$