Projectivity of good moduli spaces of quiver representations

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Joint with

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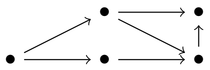

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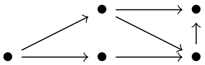

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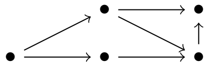

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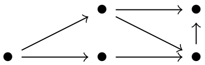

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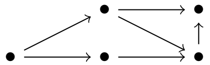

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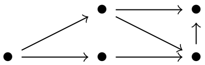

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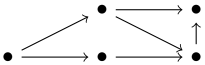

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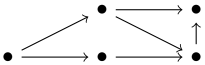

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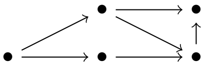

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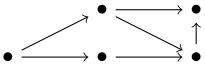

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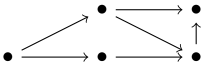

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  Goal: $\mathcal{M}_d^{\theta\text{-ss}}$ has a projective good moduli space.

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Thm: (A-HL-H) If $\mathcal{X}$ has affine diagonal, then

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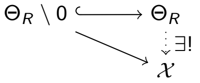

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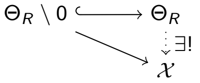

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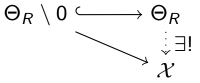

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